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Large Elastic Deformations of Isotropic Materials. IX. The Deformation of Thin Shells

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LARGE ELASTIC DEFORMATIONS OF ISOTROPIC MATERIALS

IX. THE DEFORMATION OF THIN SHELLS

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The theory of the large elastic deformation of incompressible isotropic materials is applied to problems involving thin shells. The inflation of a circular diaphragm of such a material is studied in detail. It is found that the manner in which the extension ratios and curvatures vary in the immediate neighbourhood of the pole of the inflated diaphragm can be determined analytically. However, in order to determine their variation throughout the inflated diaphragm a method of numerical integration has to be employed. Although this is, in principle, valid for any form of the stored-energy function, the calculations are carried through only for the Mooney form.

Finally, the problem of the inflation of a spherical balloon, which has already been dealt with by Green & Shield (1950), is discussed in further detail.

CONTENTS

	PAGE
1. Introduction	505
2. The equations of equilibrium for a thin sheet	507
THE INFLATION OF A CIRCULAR PLANE SHEET	508
3. Statement of the problem	508
4. The expressions for T_1 and T_2	510
5. The change of deformation in the neighbourhood of an arbitrary point—first approximation	511
6. The change of deformation in the neighbourhood of an arbitrary point—second approximation	512
7. The deformation in the neighbourhood of the pole	514
8. Deformation near the equator	518
9. The numerical solution of the problem	519
10. Comparison of theoretical and experimental results	526
A FURTHER APPLICATION OF THE THEORY	529
11. Inflation of a spherical shell	529
References	531

1. Introduction

If a thin shell of elastic material is deformed so that the principal radii of curvature are everywhere large compared with its thickness, then in setting up the equations of equilibrium and boundary conditions for the shell we can neglect the variation of stress over its thickness. The equations of equilibrium and boundary conditions then involve only the stress-resultants and stress-couples acting at each point of the shell and the applied forces.

Such equations, usually expressed with reference to a set of curvilinear co-ordinates lying in the middle surface of the deformed shell and along the normals to this surface, form the basis of the classical theory of the deformation of thin shells in which the material itself is considered to undergo only very small extensions along the principal directions. For a highly elastic material, in which the principal extension ratios may be large, it is readily seen (§ 2) that similar equations apply.

In the classical case, the stress-resultants and stress-couples can be expressed in terms of the principal extensions and curvatures of the shell, employing the classical linear stress-strain relations to describe the elastic properties of the material. For a highly elastic incompressible material, however, the expressions given by Rivlin (1948 *b*) for the components of stress in terms of the principal extension ratios and the stored-energy function for the material have to be employed in order to determine the stress-resultants.

In this paper, the inflation of a uniform circular diaphragm of incompressible highly elastic material, isotropic in its undeformed state, and clamped round its circumference, by a uniform pressure applied to one of its major surfaces is studied in detail. In this problem, as in many others involving the deformation of highly elastic shells, the effects of the stress-couples can be neglected in comparison with those of the stress-resultants, since the radii of curvature of the deformed shell are large compared with its thickness and the principal extension ratios are also large. This, and the fact that cylindrical symmetry is preserved in the deformation, produces a considerable simplification in the problem.

It has not been found possible to determine analytically the state of deformation of the diaphragm resulting from the application of a specified inflating pressure. However, if the state of deformation, defined by the extension ratios and curvatures, is known at a point of the diaphragm, then the manner in which these are changing at that point can be found analytically, for any form of the stored-energy function. This enables us to calculate the state of deformation at all points of the inflated diaphragm, by numerical integration, for any specified state of deformation at the pole. From the latter, the corresponding value of the inflating pressure is, of course, determined. The numerical integration may be carried out for any form of the stored-energy function W, but the labour of computation is greatly reduced if the Mooney form is employed. This is given by

$$W = C_1(I_1-3)+C_2(I_2-3),$$

where I_1 and I_2 are defined in terms of the principal extension ratios λ_1, λ_2 and λ_3 by

$$I_1 = \lambda_1^2+\lambda_2^2+\lambda_3^2 \quad \text{and} \quad I_2 = \lambda_1^{-2}+\lambda_2^{-2}+\lambda_3^{-2},$$

and C_1 and C_2 are physical constants for the material.

Even with this form for the stored-energy function, the computations are somewhat laborious and so have been carried out only in the cases when $C_2/C_1 = 0$ and $0{\cdot}1$. The results of the calculations are discussed in relation to the measurements made by Treloar (1944) on the inflation of a rubber diaphragm.

The method of solution that has been adopted could be employed (Adkins 1951), with only slight modification, to solve a wide variety of problems on the deformation of highly elastic shells in which both the undeformed body and the system of deforming forces have cylindrical symmetry about a common axis.

In the final section of this paper, the theory has been applied to the relatively simple problem of the inflation of a thin spherical shell. This has already been solved by Green & Shield (1950) as the limiting case of the inflation of a thick spherical shell. It is here discussed in greater detail from the point of view of the dependence on the form of the stored-energy function of the relation between inflating pressure and extent of inflation.

The results contained in this paper form part of a thesis approved by London University for the degree of Ph.D. (Adkins 1951).

2. The equations of equilibrium for a thin sheet

If a thin sheet of an elastic material is deformed, the position of any point on it may be defined with reference to a set of orthogonal curvilinear co-ordinates (α, β) which are such that the curves $\alpha = \text{const.}$ and $\beta = \text{const.}$ lie on the deformed middle surface of the sheet. Then the conditions for an element of the sheet situated at (α, β) in the deformed state to be in equilibrium are given (see, for example, Love 1927, § 331) by the six equations:

$$\left.\begin{aligned}
&\frac{\partial(T_1B)}{\partial\alpha}-\frac{\partial(S_2A)}{\partial\beta}-(r_1S_1B+r_2T_2A)+(q_1N_1B+q_2N_2A)+ABX=0,\\
&\frac{\partial(S_1B)}{\partial\alpha}+\frac{\partial(T_2A)}{\partial\beta}-(p_1N_1B+p_2N_2A)+(r_1T_1B-r_2S_2A)+ABY=0,\\
&\frac{\partial(N_1B)}{\partial\alpha}+\frac{\partial(N_2A)}{\partial\beta}-(q_1T_1B-q_2S_2A)+(p_1S_1B+p_2T_2A)+ABZ=0,\\
&\frac{\partial(H_1B)}{\partial\alpha}-\frac{\partial(G_2A)}{\partial\beta}-(r_1G_1B+r_2H_2A)+(N_2+L)\,AB=0,\\
&\frac{\partial(G_1B)}{\partial\alpha}+\frac{\partial(H_2A)}{\partial\beta}+(r_1H_1B-r_2G_2A)-(N_1-M)\,AB=0,\\
&(p_1G_1B+q_2G_2A)-(q_1H_1B-p_2H_2A)+(S_1+S_2)\,AB=0.
\end{aligned}\right\} \tag{2·1}$$

In these equations the following notation is employed:

(i) A and B are defined by the equation

$$(\delta s)^2=(A\delta\alpha)^2+(B\delta\beta)^2, \tag{2·2}$$

where δs denotes the length of the line joining the two adjacent points (α, β) and $(\alpha+\delta\alpha, \beta+\delta\beta)$.

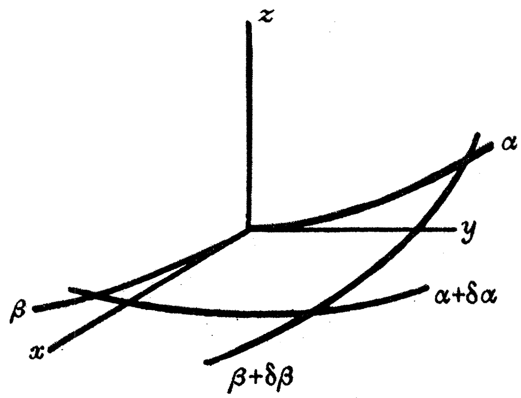

Figure 1.

(ii) Let (x, y, z) be a rectangular Cartesian co-ördinate system, having its origin at the point (α, β) and its x- and y-axes tangential to the curves $\beta = \text{const.}$ and $\alpha = \text{const.}$ respectively and in the senses of increasing α and β respectively, while its z-axis is normal to the deformed middle surface of the sheet, as shown in figure 1. Then $-T_1$, $-S_1$ and $-N_1$ are the components, in the directions x, y and z respectively, of the resultant force, acting at (α, β), per unit length of the curve $\alpha = \text{const.}$ S_2, $-T_2$ and $-N_2$ are the components, in the directions x, y and z respectively, of that per unit length of the curve $\beta = \text{const.}$

(iii) $-H_1$ and $-G_1$ are the components in the directions x and y respectively of the resultant couple, acting at (α, β), per unit length of the curve $\alpha =$ const. G_2 and $-H_2$ are analogously defined per unit length of the curve $\beta =$ const.

(iv) p_1, q_1, r_1 and p_2, q_2, r_2 are defined by the relations

$$\delta\theta_1 = p_1\delta\alpha + p_2\delta\beta, \quad \delta\theta_2 = q_1\delta\alpha + q_2\delta\beta \quad \text{and} \quad \delta\theta_3 = r_1\delta\alpha + r_2\delta\beta, \tag{2·3}$$

where $\delta\theta_1$, $\delta\theta_2$ and $\delta\theta_3$ are the components parallel to x, y and z respectively of the angle through which the co-ordinate system (x, y, z) must be rotated in order that the axes x and y shall become parallel to the tangents at $(\alpha+\delta\alpha, \beta+\delta\beta)$ to the curves $\beta =$ const. and $\alpha =$ const. which pass through that point and the z-axis shall become parallel to the normal at $(\alpha+\delta\alpha, \beta+\delta\beta)$ to the deformed middle surface of the sheet.

(v) X, Y and Z are the components parallel to the x, y and z axes respectively of the externally applied force, acting at (α, β), measured per unit area of the deformed middle surface of the sheet. L, M and 0 are the corresponding components of the externally applied couple.

The following geometrical relations obtained by Codazzi (see Love 1927, § 322) apply:

$$\left.\begin{aligned} \frac{\partial p_1}{\partial \beta} - \frac{\partial p_2}{\partial \alpha} &= q_1 r_2 - q_2 r_1, \\ \frac{\partial q_1}{\partial \beta} - \frac{\partial q_2}{\partial \alpha} &= r_1 p_2 - r_2 p_1, \\ \frac{\partial r_1}{\partial \beta} - \frac{\partial r_2}{\partial \alpha} &= p_1 q_2 - p_2 q_1, \end{aligned}\right\} \tag{2·4}$$

and

$$r_1 = -\frac{1}{B}\frac{\partial A}{\partial \beta}, \quad r_2 = \frac{1}{A}\frac{\partial B}{\partial \alpha}, \quad \frac{q_2}{B} = -\frac{p_1}{A}. \tag{2·5}$$

If the orthogonal curvilinear co-ordinate system (α, β) is so chosen that the curves $\alpha =$ const. and $\beta =$ const. are the lines of curvature of the deformed middle surface of the sheet, then

$$p_1 = q_2 = 0 \quad \text{and} \quad \frac{1}{R_1} = -\frac{q_1}{A}, \quad \frac{1}{R_2} = \frac{p_2}{B}, \tag{2·6}$$

where R_1 and R_2 are the principal radii of curvature. Introducing (2·5) and (2·6), the relations (2·4) become

$$\left.\begin{aligned} \frac{AB}{R_1 R_2} &= -\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial B}{\partial \alpha}\right) - \frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial A}{\partial \beta}\right), \\ \frac{\partial}{\partial \alpha}\left(\frac{B}{R_2}\right) &= \frac{1}{R_1}\frac{\partial B}{\partial \alpha}, \quad \frac{\partial}{\partial \beta}\left(\frac{A}{R_1}\right) = \frac{1}{R_2}\frac{\partial A}{\partial \beta}. \end{aligned}\right\} \tag{2·7}$$

THE INFLATION OF A CIRCULAR PLANE SHEET

3. Statement of the problem

We shall consider a thin flat sheet of incompressible highly elastic material which is isotropic in its undeformed state and has thickness h. This is subjected to a uniform two-dimensional extension in its own plane, in which the length of any linear element lying in the plane of the sheet undergoes an extension ratio λ_0. The sheet is then clamped in a circular clamp of radius a, as shown in figure 2a and is inflated by a uniform pressure P applied to one surface, so that it takes up a cylindrically symmetrical form such as that shown in figure 2b. It will be apparent that, as a result of the inflation, each element of area of the sheet undergoes further extension along two principal directions and that these directions which are longitudinal and latitudinal in the deformed state are radial and azimuthal respectively in the undeformed state. We shall denote the resultant extension ratios in these two directions by λ_1 and λ_2 respectively.

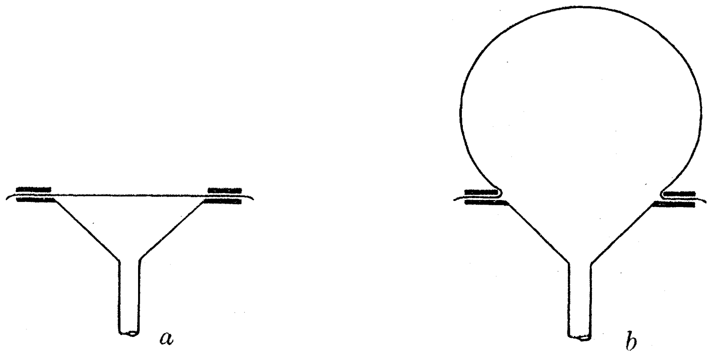

FIGURE 2.

We choose a cylindrical polar co-ordinate system (ρ, θ, z), the z-axis of which is the axis of symmetry of the deformed sheet. Then we can define any point on the deformed middle surface of the sheet by its co-ordinates ρ and θ. We can therefore take (ρ, θ) as the orthogonal curvilinear co-ordinate system (α, β) in the equations of motion (2·1). In this case we have

$$A = \mathrm{d}\xi/\mathrm{d}\rho \quad \text{and} \quad B = \rho, \tag{3·1}$$

where ξ is the length of arc measured along a line of longitude from the pole of the deformed middle surface to the point (ρ, θ).

From the symmetry of the system it is evident that

$$\left.\begin{aligned} S_1 = S_2 = N_2 = 0 \\ \text{and} \quad G_2 = H_2 = 0. \end{aligned}\right\} \tag{3·2}$$

It is also evident that the lines of curvature on the deformed middle surface are the lines of longitude and latitude on that surface so that equations (2·6) and (2·7) apply to the problem yielding, with (3·1),

$$\left.\begin{aligned} p_1 = q_2 = 0, \quad q_1 = -\frac{1}{R_1}\frac{\mathrm{d}\xi}{\mathrm{d}\rho}, \quad p_2 = \frac{\rho}{R_2}, \\ \frac{1}{R_1 R_2}\rho\left(\frac{\mathrm{d}\xi}{\mathrm{d}\rho}\right)^3 = \frac{\mathrm{d}^2\xi}{\mathrm{d}\rho^2} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\rho}{R_2}\right) = \frac{1}{R_1}. \end{aligned}\right\} \tag{3·3}$$

In deriving these equations, we note, from the symmetry of the problem, that A, B, ξ, R_1 and R_2 are independent of β, i.e. θ.

Introducing (3·1) and (3·3) into (2·5) and employing these symmetry considerations, we obtain

$$r_1 = 0 \quad \text{and} \quad r_2 = 1\Big/\frac{\mathrm{d}\xi}{\mathrm{d}\rho}. \tag{3·4}$$

Furthermore, provided that h, the thickness of the sheet, is sufficiently small compared with the radii of curvature R_1 and R_2 and the extension ratios λ_1 and λ_2 are sufficiently large, we can put

$$N_1 = 0 \quad \text{and} \quad H_1 = G_1 = 0 \tag{3·5}$$

in the equations of equilibrium (2·1).

Making the substitutions (3·1) to (3·5) in (2·1), we obtain from the first and third of these equations

$$\frac{d(T_1\rho)}{d\rho} = T_2 \quad \text{and} \quad \kappa_1 T_1 + \kappa_2 T_2 = P, \tag{3·6}$$

respectively, where κ_1 and κ_2 denote $1/R_1$ and $1/R_2$ respectively. The remaining equations of (2·1) are automatically satisfied. In carrying out this substitution we have introduced, in accordance with the conditions of the problem, $X = Y = 0$, $Z = -P$ and $L = M = 0$. We have also employed the relation $\partial T_2/\partial\theta = 0$, which arises from considerations of symmetry.

4. The expressions for T_1 and T_2

With the simplifying assumptions of § 3, we can calculate T_1 and T_2 in terms of λ_1 and λ_2 for a thin sheet of incompressible highly elastic material which is isotropic in its undeformed state.

For such a material, the stored-energy function W is a function of I_1 and I_2, given in terms of the principal extension ratios λ_1, λ_2 and λ_3 by the relations

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad \text{and} \quad I_2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2}, \tag{4·1}$$

where

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{4·2}$$

For a pure homogeneous deformation of the material, the principal components of the stress t_1, t_2 and t_3 are given (Rivlin 1948 *b*) by

$$t_i = 2\left(\lambda_i^2 \frac{\partial W}{\partial I_1} - \frac{1}{\lambda_i^2}\frac{\partial W}{\partial I_2}\right) + p \quad (i = 1, 2, 3), \tag{4·3}$$

where p is an arbitrary hydrostatic pressure. If the surface tractions on the major surfaces of the sheet are zero, then $t_3 = 0$ and equations (4·3) yield, with (4·2),

$$\left.\begin{aligned} t_1 &= 2(\lambda_1^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_2^2 \frac{\partial W}{\partial I_2}\right) \\ \text{and} \qquad t_2 &= 2(\lambda_2^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_1^2 \frac{\partial W}{\partial I_2}\right). \end{aligned}\right\} \tag{4·4}$$

In the deformation, the initial thickness h of the sheet is changed to $\lambda_3 h$. From (4·4), we obtain

$$\left.\begin{aligned} T_1 &= \lambda_3 h t_1 = 2\lambda_3 h(\lambda_1^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_2^2 \frac{\partial W}{\partial I_2}\right) \\ \text{and} \qquad T_2 &= \lambda_3 h t_2 = 2\lambda_3 h(\lambda_2^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_1^2 \frac{\partial W}{\partial I_2}\right). \end{aligned}\right\} \tag{4·5}$$

Equations (4·5) will apply with sufficient accuracy to the problem defined in § 3, in spite of the fact that the surface traction on one of the major surfaces is $-P$, since P is very small compared with T_1 and T_2.

If the point of the middle surface of the sheet, which is at (ρ, θ, z) in the deformed state, is at $(r, \theta, 0)$ in the undeformed state, so that the undeformed sheet is flat, then we readily see that

$$\lambda_1 = \frac{d\xi}{dr} \quad \text{and} \quad \lambda_2 = \frac{\rho}{r}. \tag{4·6}$$

5. The change of deformation in the neighbourhood of an arbitrary point—first approximation

If λ_1, λ_2, κ_1 and κ_2 are known at any point which was at r in the undeformed state, then their derivatives $d\lambda_1/dr$, $d\lambda_2/dr$, etc., can be found in terms of them and the values of λ_1, λ_2, etc., at an adjacent point, which was at $(r+\Delta r)$ in the undeformed state, can be found to a first approximation from relations of the type

$$[\lambda_1]_{r+\Delta r} = [\lambda_1]_r + [d\lambda_1/dr]_r \Delta r. \tag{5·1}$$

The equations governing the deformation are (3·6) and the last two of equations (3·3). These may be rewritten as

$$\left.\begin{aligned} \frac{d}{dr}(T_1\rho) &= T_2\frac{d\rho}{dr},\\ \kappa_1 T_1+\kappa_2 T_2 &= P,\\ \frac{d}{dr}(\kappa_2\rho) &= \kappa_1\frac{d\rho}{dr}\\ \text{and}\qquad \kappa_1\kappa_2\rho\lambda_1^3 &= \frac{d\rho}{dr}\frac{d\lambda_1}{dr}-\lambda_1\frac{d^2\rho}{dr^2}. \end{aligned}\right\} \tag{5·2}$$

In deriving the last of these equations, the first of the relations (4·6) must be employed.

The first and third of equations (5·2) yield

$$\left.\begin{aligned} \frac{dT_1}{dr} &= -\frac{1}{\rho}\frac{d\rho}{dr}(T_1-T_2)\\ \text{and}\qquad \frac{d\kappa_2}{dr} &= \frac{1}{\rho}\frac{d\rho}{dr}(\kappa_1-\kappa_2). \end{aligned}\right\} \tag{5·3}$$

From the last two of equations (3·3) or directly from geometrical considerations, it can be shown that

$$\kappa_2 = \frac{1}{\rho}\left[1-\left(\frac{d\rho}{d\xi}\right)^2\right]^{\frac{1}{2}}. \tag{5·4}$$

Since $d\rho/d\xi = (d\rho/dr)/(d\xi/dr)$, we have from (5·4) and the first of equations (4·6)

$$\frac{d\rho}{dr} = \lambda_1(1-\kappa_2^2\rho^2)^{\frac{1}{2}}. \tag{5·5}$$

With the second of equations (4·6), (5·5) yields

$$\frac{d\rho}{dr} = \lambda_1(1-\kappa_2^2\lambda_2^2r^2)^{\frac{1}{2}}. \tag{5·6}$$

Substituting in (5·3) from (5·6), (4·5) and (4·6), we obtain

$$\left.\begin{aligned} \frac{dT_1}{dr} &= -\frac{2h}{\lambda_2^2 r}(1-\kappa_2^2\lambda_2^2r^2)^{\frac{1}{2}}(\lambda_1^2-\lambda_2^2)\left(\frac{\partial W}{\partial I_1}+\lambda_3^2\frac{\partial W}{\partial I_2}\right)\\ \text{and}\qquad \frac{d\kappa_2}{dr} &= \frac{\lambda_1}{\lambda_2 r}(1-\kappa_2^2\lambda_2^2r^2)^{\frac{1}{2}}(\kappa_1-\kappa_2). \end{aligned}\right\} \tag{5·7}$$

Differentiating the second of equations (4·6) and substituting for $d\rho/dr$ from (5·6), we obtain

$$\frac{d\lambda_2}{dr} = \frac{1}{r}[\lambda_1(1-\kappa_2^2\lambda_2^2r^2)^{\frac{1}{2}} - \lambda_2]. \tag{5·8}$$

Differentiating the first of equations (4·5), we obtain

$$\frac{1}{2h}\frac{dT_1}{dr} = \left[(\lambda_1^2+3\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_2^2\frac{\partial W}{\partial I_2}\right)+2(\lambda_1^2-\lambda_3^2)^2\left(\frac{\partial^2 W}{\partial I_1^2}+2\lambda_2^2\frac{\partial^2 W}{\partial I_1\partial I_2}+\lambda_2^4\frac{\partial^2 W}{\partial I_2^2}\right)\right]\frac{\lambda_3}{\lambda_1}\frac{\partial\lambda_1}{dr}$$
$$+\left[(3\lambda_3^2-\lambda_1^2)\frac{\partial W}{\partial I_1}+(\lambda_1^2+\lambda_3^2)\lambda_2^2\frac{\partial W}{\partial I_2}+2(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)\right.$$
$$\left.\times\left\{\frac{\partial^2 W}{\partial I_1^2}+(\lambda_1^2+\lambda_2^2)\frac{\partial^2 W}{\partial I_1\partial I_2}+\lambda_1^2\lambda_2^2\frac{\partial^2 W}{\partial I_2^2}\right\}\right]\frac{\lambda_3}{\lambda_2}\frac{\partial\lambda_2}{dr}. \tag{5·9}$$

dT_2/dr is given by an equation similar to (5·9) in which λ_1 and λ_2 are interchanged.

Since dT_1/dr is given by the first of equations (5·7) and $d\lambda_2/dr$ by (5·8), we can calculate $d\lambda_1/dr$, from (5·9), in terms of λ_1, λ_2, κ_2 and r.

From this expression for $d\lambda_1/dr$ and the expression (5·8) for $d\lambda_2/dr$, dT_2/dr can be calculated by means of an expression similar to (5·9). Now differentiating the second of equations (5·2), we obtain

$$\kappa_1\frac{dT_1}{dr}+\kappa_2\frac{dT_2}{dr}+T_1\frac{d\kappa_1}{dr}+T_2\frac{d\kappa_2}{dr}=0. \tag{5·10}$$

It has already been seen how expressions for dT_1/dr, dT_2/dr and $d\kappa_2/dr$ in terms of λ_1, λ_2, κ_1, κ_2 and r can be obtained. Thus, equation (5·10) provides a relation by means of which such an expression can be obtained for $d\kappa_1/dr$.

If the stored-energy function W has the Mooney form, so that

$$W = C_1(I_1-3)+C_2(I_2-3), \tag{5·11}$$

where C_1 and C_2 are physical constants, then the expressions (4·5) for T_1 and T_2 become

$$\left.\begin{aligned} T_1 &= 2\lambda_3 h(\lambda_1^2-\lambda_3^2)(C_1+\lambda_2^2C_2) \\ \text{and}\quad T_2 &= 2\lambda_3 h(\lambda_2^2-\lambda_3^2)(C_1+\lambda_1^2C_2).\end{aligned}\right\} \tag{5·12}$$

From (5·9) and a similar equation for dT_2/dr, we obtain the simpler expressions

$$\left.\begin{aligned} \frac{1}{2h}\frac{dT_1}{dr} &= [(\lambda_1^2+3\lambda_3^2)(C_1+\lambda_2^2C_2)]\frac{\lambda_3}{\lambda_1}\frac{d\lambda_1}{dr}+[(3\lambda_3^2-\lambda_1^2)C_1+(\lambda_1^2+\lambda_3^2)\lambda_2^2C_2]\frac{\lambda_3}{\lambda_2}\frac{d\lambda_2}{dr} \\ \text{and}\quad \frac{1}{2h}\frac{dT_2}{dr} &= [(\lambda_2^2+3\lambda_3^2)(C_1+\lambda_1^2C_2)]\frac{\lambda_3}{\lambda_2}\frac{d\lambda_2}{dr}+[(3\lambda_3^2-\lambda_2^2)C_1+(\lambda_2^2+\lambda_3^2)\lambda_1^2C_2]\frac{\lambda_3}{\lambda_1}\frac{d\lambda_1}{dr}.\end{aligned}\right\} \tag{5·13}$$

6. The change of deformation in the neighbourhood of an arbitrary point—second approximation

If λ_1, λ_2, κ_1 and κ_2 are known at any point, which was at r in the undeformed state, then their values at an adjacent point, which was at $r+\Delta r$ in the undeformed state, can be found to a second approximation from relations of the type

$$[\lambda_1]_{r+\Delta r} = [\lambda_1]_r+[d\lambda_1/dr]_r\,\Delta r+\tfrac{1}{2}[d^2\lambda_1/dr^2]_r\,(\Delta r)^2. \tag{6·1}$$

It has been seen in § 5 how expressions for $\mathrm{d}\lambda_1/\mathrm{d}r$, $\mathrm{d}\lambda_2/\mathrm{d}r$, $\mathrm{d}\kappa_1/\mathrm{d}r$ and $\mathrm{d}\kappa_2/\mathrm{d}r$ in terms of λ_1, λ_2, κ_1, κ_2 and r can be found. In this section it will be shown how expressions for $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$, $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$, $\mathrm{d}^2\kappa_1/\mathrm{d}r^2$ and $\mathrm{d}^2\kappa_2/\mathrm{d}r^2$ can be found in terms of these quantities.

Differentiating the second of equations (4·6) twice and each of equations (5·3) once, we obtain respectively

$$\left.\begin{aligned} r\frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2} &= \frac{\mathrm{d}^2\rho}{\mathrm{d}r^2} - 2\frac{\mathrm{d}\lambda_2}{\mathrm{d}r}, \\ \lambda_2 r\frac{\mathrm{d}^2T_1}{\mathrm{d}r^2} &= -\frac{\mathrm{d}\rho}{\mathrm{d}r}\left(2\frac{\mathrm{d}T_1}{\mathrm{d}r} - \frac{\mathrm{d}T_2}{\mathrm{d}r}\right) - \frac{\mathrm{d}^2\rho}{\mathrm{d}r^2}(T_1 - T_2) \\ \text{and}\qquad \lambda_2 r\frac{\mathrm{d}^2\kappa_2}{\mathrm{d}r^2} &= \frac{\mathrm{d}\rho}{\mathrm{d}r}\left(\frac{\mathrm{d}\kappa_1}{\mathrm{d}r} - 2\frac{\mathrm{d}\kappa_2}{\mathrm{d}r}\right) + \frac{\mathrm{d}^2\rho}{\mathrm{d}r^2}(\kappa_1 - \kappa_2). \end{aligned}\right\} \qquad (6{\cdot}2)$$

In these equations the second of the relations (4·6), $\rho = \lambda_2 r$, has been used.

Differentiating equation (5·9) we obtain

$$\begin{aligned} \frac{1}{2h}\frac{\mathrm{d}^2T_1}{\mathrm{d}r^2} = {} & \left[(\lambda_1^2+3\lambda_3^2)\left(\frac{\partial}{\partial I_1}+\lambda_2^2\frac{\partial}{\partial I_2}\right)+2(\lambda_1^2-\lambda_3^2)^2\left(\frac{\partial}{\partial I_1}+\lambda_2^2\frac{\partial}{\partial I_2}\right)^2\right]W\frac{\lambda_3}{\lambda_1}\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2} \\ & -\left[(\lambda_1^2-3\lambda_3^2)\frac{\partial}{\partial I_1}-(\lambda_1^2+\lambda_3^2)\lambda_2^2\frac{\partial}{\partial I_2}\right. \\ & \left.\quad -2(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)\left(\frac{\partial}{\partial I_1}+\lambda_1^2\frac{\partial}{\partial I_2}\right)\left(\frac{\partial}{\partial I_1}+\lambda_2^2\frac{\partial}{\partial I_2}\right)\right]W\frac{\lambda_3}{\lambda_2}\frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2} \\ & -2\left[6\lambda_3^2\left(\frac{\partial}{\partial I_1}+\lambda_2^2\frac{\partial}{\partial I_2}\right)-3(\lambda_1^2-\lambda_3^2)(\lambda_1^2+3\lambda_3^2)\left(\frac{\partial}{\partial I_1}+\lambda_2^2\frac{\partial}{\partial I_2}\right)^2\right. \\ & \left.\quad -2(\lambda_1^2-\lambda_3^2)^3\left(\frac{\partial}{\partial I_1}+\lambda_2^2\frac{\partial}{\partial I_2}\right)^3\right]W\frac{\lambda_3}{\lambda_1^2}\left(\frac{\mathrm{d}\lambda_1}{\mathrm{d}r}\right)^2 \\ & +2\left[(\lambda_1^2-6\lambda_3^2)\frac{\partial}{\partial I_1}-\lambda_2^2\lambda_3^2\frac{\partial}{\partial I_2}+\{\lambda_3^2(5\lambda_1^2+5\lambda_2^2-9\lambda_3^2)-\lambda_1^2\lambda_2^2\}\frac{\partial^2}{\partial I_1^2}\right. \\ & \quad +\{6+\lambda_1^4(5\lambda_3^2-\lambda_2^2)+\lambda_2^4(3\lambda_1^2+\lambda_3^2)-\lambda_3^4(9\lambda_1^2+5\lambda_2^2)\}\frac{\partial^2}{\partial I_1\partial I_2} \\ & \quad +(\lambda_1^2+\lambda_2^2-5\lambda_3^2+3\lambda_1^4\lambda_2^4)\frac{\partial^2}{\partial I_2^2} \\ & \left.\quad +2(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)^2\left(\frac{\partial}{\partial I_1}+\lambda_1^2\frac{\partial}{\partial I_2}\right)^2\left(\frac{\partial}{\partial I_1}+\lambda_2^2\frac{\partial}{\partial I_2}\right)\right]W\frac{\lambda_3}{\lambda_2^2}\left(\frac{\mathrm{d}\lambda_2}{\mathrm{d}r}\right)^2 \\ & -2\left[(\lambda_1^2+9\lambda_3^2)\frac{\partial}{\partial I_1}-(\lambda_1^2-3\lambda_3^2)\lambda_2^2\frac{\partial}{\partial I_2}+2\{\lambda_1^2(\lambda_1^2-\lambda_2^2-5\lambda_3^2)-\lambda_3^2(3\lambda_2^2-8\lambda_3^2)\}\frac{\partial^2}{\partial I_1^2}\right. \\ & \quad -2(\lambda_1^2+3\lambda_3^2)\{\lambda_2^2(\lambda_2^2-3\lambda_3^2)+\lambda_1^2(3\lambda_2^2-\lambda_3^2)\}\frac{\partial^2}{\partial I_1\partial I_2} \\ & \quad +2\{\lambda_1^2-\lambda_2^2+3\lambda_3^2-\lambda_2^4(4\lambda_1^4-\lambda_3^4)\}\frac{\partial^2}{\partial I_2^2} \\ & \left.\quad -4(\lambda_1^2-\lambda_3^2)^2(\lambda_2^2-\lambda_3^2)\left(\frac{\partial}{\partial I_1}+\lambda_2^2\frac{\partial}{\partial I_2}\right)^2\left(\frac{\partial}{\partial I_1}+\lambda_1^2\frac{\partial}{\partial I_2}\right)\right]W\lambda_3^2\frac{\mathrm{d}\lambda_1}{\mathrm{d}r}\frac{\mathrm{d}\lambda_2}{\mathrm{d}r}. \end{aligned} \qquad (6{\cdot}3)$$

$\mathrm{d}^2T_2/\mathrm{d}r^2$ is given by a similar expression in which λ_1 and λ_2 are interchanged.

Again, differentiating equation (5·10), we obtain

$$\frac{d^2\kappa_1}{dr^2} = -\frac{1}{T_1}\left[2\frac{d\kappa_1}{dr}\frac{dT_1}{dr} + \kappa_1\frac{d^2T_1}{dr^2} + T_2\frac{d^2\kappa_2}{dr^2} + 2\frac{d\kappa_2}{dr}\frac{dT_2}{dr} + \kappa_2\frac{d^2T_2}{dr^2}\right]. \tag{6·4}$$

In § 5 it has been seen that the first derivatives of ρ, λ_1, λ_2, κ_1, κ_2, T_1 and T_2 can be expressed in terms of λ_1, λ_2, κ_1, κ_2 and r. From the last of equations (5·2), $d^2\rho/dr^2$ can be expressed in terms of these quantities, and thus, from (6·2) $d^2\lambda_2/dr^2$, d^2T_1/dr^2 and $d^2\kappa_2/dr^2$ can be so expressed. From (6·3) an expression in terms of λ_1, λ_2, etc., can be obtained for $d^2\lambda_1/dr^2$, and from the expression for d^2T_2/dr^2, analogous with (6·3) and equation (6·4), an expression for $d^2\kappa_1/dr^2$ in terms of λ_1, λ_2, etc., can be obtained.

It is evident that by continued differentiation of equations (5·2), (4·5) and the second of (4·6) we may obtain expressions for the third and higher order derivatives of λ_1, λ_2, κ_1, κ_2, T_1, T_2 and ρ, and by taking additional terms of the Taylor series obtain more accurate expressions for the quantities λ_1, λ_2, κ_1 and κ_2 at the point $(r+\Delta r)$.

If the stored-energy function W has the Mooney form (5·11), then equation (6·3) yields the much simpler expression

$$\begin{aligned}\frac{1}{2h}\frac{d^2T_1}{dr^2} &= [(\lambda_1^2+3\lambda_3^2)(C_1+\lambda_2^2C_2)]\frac{\lambda_3}{\lambda_1}\frac{d^2\lambda_1}{dr^2}\\ &+[(3\lambda_3^2-\lambda_1^2)C_1+(\lambda_1^2+\lambda_3^2)\lambda_2^2C_2]\frac{\lambda_3}{\lambda_2}\frac{d^2\lambda_2}{dr^2}-12(C_1+\lambda_2^2C_2)\frac{\lambda_3^3}{\lambda_1^2}\left(\frac{d\lambda_1}{dr}\right)^2\\ &+2[-(\lambda_1^2+9\lambda_3^2)C_1+\lambda_2^2(\lambda_1^2-3\lambda_3^2)C_2]\lambda_3^2\frac{d\lambda_1}{dr}\frac{d\lambda_2}{dr}\\ &+2\lambda_3^3[\lambda_1^2(\lambda_1^2-6\lambda_3^2)C_1-C_2]\left(\frac{d\lambda_2}{dr}\right)^2.\end{aligned} \tag{6·5}$$

In a similar manner we obtain

$$\begin{aligned}\frac{1}{2h}\frac{d^2T_2}{dr^2} &= [(\lambda_2^2+3\lambda_3^2)(C_1+\lambda_1^2C_2)]\frac{\lambda_3}{\lambda_2}\frac{d^2\lambda_2}{dr^2}\\ &+[(3\lambda_3^2-\lambda_2^2)C_1+(\lambda_2^2+\lambda_3^2)\lambda_1^2C_2]\frac{\lambda_3}{\lambda_1}\frac{d^2\lambda_1}{dr^2}-12\lambda_3^2(C_1+\lambda_1^2C_2)\frac{\lambda_3}{\lambda_2^2}\left(\frac{d\lambda_2}{dr}\right)^2\\ &+2[-(\lambda_2^2+9\lambda_3^2)C_1+\lambda_1^2(\lambda_2^2-3\lambda_3^2)C_2]\lambda_3^2\frac{d\lambda_1}{dr}\frac{d\lambda_2}{dr}\\ &+2\lambda_3^3[\lambda_2^2(\lambda_2^2-6\lambda_3^2)C_1-C_2]\left(\frac{d\lambda_1}{dr}\right)^2.\end{aligned} \tag{6·6}$$

7. The deformation in the neighbourhood of the pole

At the pole, we see from symmetry that

$$\kappa_1=\kappa_2=\kappa \text{ (say)}, \quad \lambda_1=\lambda_2=\lambda \text{ (say)} \quad \text{and} \quad T_1=T_2=T \text{ (say)}. \tag{7·1}$$

From (4·5) we have, at the pole,

$$T = 2h\left(1-\frac{1}{\lambda^6}\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right),$$

where

$$I_1 = 2\lambda^2+\frac{1}{\lambda^4} \quad \text{and} \quad I_2 = \frac{2}{\lambda^2}+\lambda^4. \tag{7·2}$$

From the second of equations (5·2) we have

$$T=\frac{P}{2\kappa}. \tag{7·3}$$

From considerations of symmetry, we see that κ_1, κ_2, λ_1, λ_2, T_1 and T_2 must be even functions of r and that ρ must be an odd function of r, so that, at the pole,

$$\left.\begin{aligned}\frac{d\kappa_1}{dr}=\frac{d\kappa_2}{dr}=\frac{d\lambda_1}{dr}=\frac{d\lambda_2}{dr}=\frac{dT_1}{dr}=\frac{dT_2}{dr}=0\\ \text{and}\qquad \frac{d^2\rho}{dr^2}=0.\end{aligned}\right\} \tag{7·4}$$

We have also, at the pole,

$$\frac{d\rho}{dr}=\lambda. \tag{7·5}$$

$d^2\lambda_1/dr^2$, $d^2\lambda_2/dr^2$, etc., cannot be determined at the pole by the relations of § 6, since equations (6·2) vanish identically if the conditions (7·1), (7·4) and $r=0$, which apply there, are introduced. They may, however, be obtained in the following manner.

Differentiating the first of equations (5·2) twice with respect to r, writing $\rho=0$ and introducing the relations (7·4) and (7·1), we obtain at the pole

$$3\frac{d^2T_1}{dr^2}=\frac{d^2T_2}{dr^2}. \tag{7·6}$$

In a similar manner we see, from the third of equations (5·2), that at the pole

$$3\frac{d^2\kappa_2}{dr^2}=\frac{d^2\kappa_1}{dr^2}. \tag{7·7}$$

From (6·4), (7·1), (7·4), (7·6) and (7·7) we obtain

$$T\frac{d^2\kappa_2}{dr^2}+\kappa\frac{d^2T_1}{dr^2}=0. \tag{7·8}$$

Differentiating the last of equations (5·2) with respect to r and introducing the relations (7·1), (7·4) and (7·5), we obtain at the pole, where $r=0$,

$$\frac{d^3\rho}{dr^3}-\frac{d^2\lambda_1}{dr^2}=-\kappa^2\lambda^3. \tag{7·9}$$

Again, differentiating the second of equations (4·6) thrice with respect to r, we obtain at the pole, where $r=0$,

$$\frac{d^3\rho}{dr^3}=3\frac{d^2\lambda_2}{dr^2}. \tag{7·10}$$

Equations (7·9) and (7·10) yield

$$3\frac{d^2\lambda_2}{dr^2}-\frac{d^2\lambda_1}{dr^2}=-\kappa^2\lambda^3. \tag{7·11}$$

Introducing into (6·3) the relations (7·4) and (7·1) we obtain at the pole

$$\begin{aligned}\frac{1}{2h}\frac{d^2T_1}{dr^2}=&\left(\lambda^2+\frac{3}{\lambda^4}\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)\frac{1}{\lambda^3}\frac{d^2\lambda_1}{dr^2}\\&+\left[\left(\frac{3}{\lambda^4}-\lambda^2\right)\frac{\partial W}{\partial I_1}+\left(\lambda^4+\frac{1}{\lambda^2}\right)\frac{\partial W}{\partial I_2}\right]\frac{1}{\lambda^3}\frac{d^2\lambda_2}{dr^2}\\&+2\left(\lambda^2-\frac{1}{\lambda^4}\right)^2\left(\frac{\partial}{\partial I_1}+\lambda^2\frac{\partial}{\partial I_2}\right)^2W\frac{1}{\lambda^3}\left(\frac{d^2\lambda_1}{dr^2}+\frac{d^2\lambda_2}{dr^2}\right).\end{aligned} \tag{7·12}$$

Similarly, we can obtain

$$\frac{1}{2h}\frac{\mathrm{d}^2T_2}{\mathrm{d}r^2}=\left(\lambda^2+\frac{3}{\lambda^4}\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)\frac{1}{\lambda^3}\frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2}$$
$$+\left[\left(\frac{3}{\lambda^4}-\lambda^2\right)\frac{\partial W}{\partial I_1}+\left(\lambda^4+\frac{1}{\lambda^2}\right)\frac{\partial W}{\partial I_2}\right]\frac{1}{\lambda^3}\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2}$$
$$+2\left(\lambda^2-\frac{1}{\lambda^4}\right)^2\left(\frac{\partial}{\partial I_1}+\lambda^2\frac{\partial}{\partial I_2}\right)^2W\frac{1}{\lambda^3}\left(\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2}+\frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2}\right). \qquad (7\cdot13)$$

Introducing the expressions (7·12) and (7·13) into (7·6), we obtain at the pole

$$\frac{\partial W}{\partial I_1}\left[\left(2+\frac{3}{\lambda^6}\right)\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2}-\left(2-\frac{3}{\lambda^6}\right)\frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2}\right]+\lambda^2\frac{\partial W}{\partial I_2}\left[\left(1+\frac{4}{\lambda^6}\right)\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2}+\frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2}\right]$$
$$+2\lambda^2\left(1-\frac{1}{\lambda^6}\right)^2\left(\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2}+\frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2}\right)\left(\frac{\partial}{\partial I_1}+\lambda^2\frac{\partial}{\partial I_2}\right)^2W=0. \qquad (7\cdot14)$$

Solving equations (7·14) and (7·11) for the values of $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ at the pole, we obtain

$$\left.\begin{aligned}\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2}&=\frac{-\tfrac{1}{4}\kappa^2\lambda^3\left[\frac{\partial W}{\partial I_1}\left(2-\frac{3}{\lambda^6}\right)-\lambda^2\frac{\partial W}{\partial I_2}-2\lambda^2\left(1-\frac{1}{\lambda^6}\right)^2A\right]}{\left(1+\frac{3}{\lambda^6}\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)+2\lambda^2\left(1-\frac{1}{\lambda^6}\right)^2A}\\ \text{and}\qquad\frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2}&=\frac{-\tfrac{1}{4}\kappa^2\lambda^3\left[\frac{\partial W}{\partial I_1}\left(2+\frac{3}{\lambda^6}\right)+\lambda^2\left(1+\frac{4}{\lambda^6}\right)\frac{\partial W}{\partial I_2}+2\lambda^2\left(1-\frac{1}{\lambda^6}\right)^2A\right]}{\left(1+\frac{3}{\lambda^6}\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)+2\lambda^2\left(1-\frac{1}{\lambda^6}\right)^2A},\end{aligned}\right\} \qquad (7\cdot15)$$

where
$$A=\frac{\partial^2W}{\partial I_1^2}+2\lambda^2\frac{\partial^2W}{\partial I_1\partial I_2}+\lambda^4\frac{\partial^2W}{\partial I_2^2}. \qquad (7\cdot16)$$

From these expressions for $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$, $\mathrm{d}^2T_1/\mathrm{d}r^2$ can be found by means of equation (7·12). We obtain

$$\frac{1}{2h}\frac{\mathrm{d}^2T_1}{\mathrm{d}r^2}=\frac{\begin{aligned}-\kappa^2\lambda^2\left[\frac{3}{\lambda^6}\left(\frac{\partial W}{\partial I_1}\right)^2+\left(1+\frac{2}{\lambda^6}+\frac{3}{\lambda^{12}}\right)\lambda^2\frac{\partial W}{\partial I_1}\frac{\partial W}{\partial I_2}\right.\\ \left.+\frac{1}{\lambda^2}\left(1+\frac{2}{\lambda^6}\right)\left(\frac{\partial W}{\partial I_2}\right)^2+2\lambda^2\left(1-\frac{1}{\lambda^6}\right)^2\left(\frac{\partial W}{\partial I_1}+\frac{1}{\lambda^4}\frac{\partial W}{\partial I_2}\right)A\right]\end{aligned}}{2\left[\left(1+\frac{3}{\lambda^6}\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)+2\lambda^2\left(1+\frac{1}{\lambda^6}\right)^2A\right]}. \qquad (7\cdot17)$$

Equations (7·6) to (7·8) then yield expressions for $\mathrm{d}^2T_2/\mathrm{d}r^2$, $\mathrm{d}^2\kappa_1/\mathrm{d}r^2$ and $\mathrm{d}^2\kappa_2/\mathrm{d}r^2$.

Since $\mathrm{d}\lambda_1/\mathrm{d}r=\mathrm{d}\lambda_2/\mathrm{d}r=0$, the values of $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ determine the shapes in the neighbourhood of the pole of the λ_1 against r and λ_2 against r curves respectively. We bear in mind that $\lambda_1=\lambda_2=\lambda$ at the pole (i.e. when $r=0$), so that if the two curves are plotted on a single graph they have a common point where $r=0$. We shall consider the forms of the two curves in the neighbourhood of this point.

If the highly elastic material considered is an incompressible, neo-Hookean material, so that W is given (Rivlin 1948*a*) by

$$W=C(I_1-3), \qquad (7\cdot18)$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

where C is a constant, equations (7·15) become

$$\left.\begin{aligned} \frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2} &= -\frac{\frac{1}{4}\kappa^2\lambda^3\left(2-\dfrac{3}{\lambda^6}\right)}{1+\dfrac{3}{\lambda^6}} \\ \text{and}\qquad \frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2} &= -\frac{\frac{1}{4}\kappa^2\lambda^3\left(2+\dfrac{3}{\lambda^6}\right)}{1+\dfrac{3}{\lambda^6}}. \end{aligned}\right\} \tag{7·19}$$

λ is always greater than unity for the type of deformation envisaged. When $\lambda^6 < \frac{3}{2}$, we see that $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ is positive and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ is negative, so that in the neighbourhood of $r = 0$, λ_1 increases and λ_2 decreases as r increases. When $\lambda^6 > \frac{3}{2}$, $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ are both negative, so that, in the immediate neighbourhood of the pole, both λ_1 and λ_2 decrease with increase of r. When $\lambda^6 \gg \frac{3}{2}$, $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ become nearly equal. In practical cases for which it is permissible to neglect the couples G_1 and H_1, so that the basic equations of the theory are valid, we will generally be concerned with values of λ such that $\lambda^6 \gg \frac{3}{2}$.

We shall now consider the case when W has the Mooney form given by

$$W = C_1(I_1-3)+C_2(I_2-3), \tag{7·20}$$

where C_1 and C_2 are positive constants. Then equations (7·15) become

$$\left.\begin{aligned} \frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2} &= \frac{-\frac{1}{4}\kappa^2\lambda^3\left[\left(2-\dfrac{3}{\lambda^6}\right)C_1-\lambda^2C_2\right]}{\left(1+\dfrac{3}{\lambda^6}\right)(C_1+\lambda^2C_2)} \\ \text{and}\qquad \frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2} &= \frac{-\frac{1}{4}\kappa^2\lambda^3\left[\left(2+\dfrac{3}{\lambda^6}\right)C_1+\lambda^2\left(1+\dfrac{4}{\lambda^6}\right)C_2\right]}{\left(1+\dfrac{3}{\lambda^6}\right)(C_1+\lambda^2C_2)}. \end{aligned}\right\} \tag{7·21}$$

We see that whereas $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ is always negative, as in the case of an incompressible neo-Hookean material, $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ is positive for values of $\lambda^6 < \frac{3}{2}$, may then become negative for larger values of λ provided that C_2/C_1 is sufficiently small and becomes positive again for still greater values of λ.

The presence in the expression for the stored-energy function W of terms of higher degree than the first in (I_1-3) and (I_2-3) may have a profound effect on the values of $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$, even though their coefficients are small compared with those of the first degree terms. This can be seen by noting that for large values of λ, the terms involving $\partial^2W/\partial I_2^2$ in the expressions (7·15) for $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ are multiplied by $2\lambda^6$ and the terms in $\partial^2W/\partial I_1\,\partial I_2$ by $4\lambda^4$. A non-zero value for $\partial^2W/\partial I_2^2$ or $\partial^2W/\partial I_1\,\partial I_2$ will therefore have the effect of making the terms contained in A the controlling terms in determining the values of $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ when λ is very large, giving positive and negative values for $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ respectively for such values of λ, these values becoming approximately equal in magnitude for sufficiently large values of λ. Similar considerations apply to $\mathrm{d}^2T_1/\mathrm{d}r^2$.

From equations (7·12) and (7·14) we may write $\mathrm{d}^2T_1/\mathrm{d}r^2$ in the form

$$\frac{\mathrm{d}^2T_1}{\mathrm{d}r^2}=\frac{2h}{\lambda}\left(\frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2}-\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2}\right)\left(\frac{\partial W}{\partial I_1}+\frac{1}{\lambda^4}\frac{\partial W}{\partial I_2}\right). \tag{7·22}$$

With equation (7·8) this yields

$$\frac{T}{\kappa}\frac{\mathrm{d}^2\kappa_2}{\mathrm{d}r^2}=-\frac{2h}{\lambda}\left(\frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2}-\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2}\right)\left(\frac{\partial W}{\partial I_1}+\frac{1}{\lambda^4}\frac{\partial W}{\partial I_2}\right). \tag{7·23}$$

From (7·23) we should expect that for given values of T and κ, $\mathrm{d}^2\kappa_2/\mathrm{d}r^2$ would be most nearly zero when $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ are of the same sign and approximately equal. This situation arises in the case of the incompressible neo-Hookean material when λ is large. The smaller $\mathrm{d}^2\kappa_2/\mathrm{d}r^2$, the larger will be the region in the neighbourhood of the pole of the inflated sheet that is substantially spherical.

It is evident that by continued differentiations of (5·2), (4·5) and the second of equations (4·6) and the introduction of the symmetry conditions at the pole we may derive expressions for the fourth and higher order derivatives of λ_1, λ_2, κ_1, κ_2, T_1 and T_2. If the material has the neo-Hookean form of stored-energy function, the expressions for the fourth derivatives assume reasonably simple forms (Adkins 1951), but for more general forms of W the expressions become extremely cumbersome.

8. Deformation near the equator

At the equator $\mathrm{d}\rho/\mathrm{d}r=0$. With the second of equations (4·6), this yields

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}r}=-\frac{\lambda_2}{r}. \tag{8·1}$$

With (5·8), (8·1) yields
$$\kappa_2\lambda_2 r=1. \tag{8·2}$$

Introducing $\mathrm{d}\rho/\mathrm{d}r=0$ and the relation (8·2) into the first, third and last of equations (5·2), they yield

$$\frac{\mathrm{d}T_1}{\mathrm{d}r}=0,\quad \frac{\mathrm{d}\kappa_2}{\mathrm{d}r}=0\quad\text{and}\quad \frac{\mathrm{d}^2\rho}{\mathrm{d}r^2}=-\kappa_1\lambda_1^2 \tag{8·3}$$

respectively.

From (5·9), (8·1) and the first of equations (8·3), we obtain

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}r}=\frac{\dfrac{\lambda_1}{r}\left[(3\lambda_3^2-\lambda_1^2)\dfrac{\partial W}{\partial I_1}+(\lambda_1^2+\lambda_3^2)\,\lambda_2^2\dfrac{\partial W}{\partial I_2}+2(\lambda_1^2-\lambda_3^2)\,(\lambda_2^2-\lambda_3^2)\left\{\dfrac{\partial^2W}{\partial I_1^2}+(\lambda_1^2+\lambda_2^2)\dfrac{\partial^2W}{\partial I_1\partial I_2}+\lambda_1^2\lambda_2^2\dfrac{\partial^2W}{\partial I_2^2}\right\}\right]}{(\lambda_1^2+3\lambda_3^2)\left(\dfrac{\partial W}{\partial I_1}+\lambda_2^2\dfrac{\partial W}{\partial I_2}\right)+2(\lambda_1^2-\lambda_3^2)^2\left(\dfrac{\partial^2W}{\partial I_1^2}+2\lambda_2^2\dfrac{\partial^2W}{\partial I_1\partial I_2}+\lambda_2^4\dfrac{\partial^2W}{\partial I_2^2}\right)}. \tag{8·4}$$

Substituting from (8·1) and (8·4) for $\mathrm{d}\lambda_1/\mathrm{d}r$ and $\mathrm{d}\lambda_2/\mathrm{d}r$ respectively in the expression for $\mathrm{d}T_2/\mathrm{d}r$ similar to (5·9), we can obtain $\mathrm{d}T_2/\mathrm{d}r$ in terms of λ_1 and λ_2.

Also, from equations (5·10), (8·2) and (8·3), we obtain

$$\frac{\mathrm{d}\kappa_1}{\mathrm{d}r}=-\frac{1}{\lambda_2 rT_1}\frac{\mathrm{d}T_2}{\mathrm{d}r}. \tag{8·5}$$

Since it has already been seen how T_1 and $\mathrm{d}T_2/\mathrm{d}r$ can be expressed in terms of λ_1, λ_2 and r, (8·5) enables us to express $\mathrm{d}\kappa_1/\mathrm{d}r$ in these terms.

Having seen how the first derivatives of λ_1, λ_2, κ_1, κ_2, T_1 and T_2 at the equator of the inflated sheet can be expressed in terms of λ_1, λ_2 and r, we now proceed to obtain expressions for the second derivatives in terms of λ_1, λ_2, r and κ_1.

Introducing $\mathrm{d}\rho/\mathrm{d}r = 0$ and the relations (8·3) and (8·1) into equations (6·2), we obtain

$$\left.\begin{aligned} \frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2} &= \frac{2\lambda_2}{r^2} - \frac{\kappa_1\lambda_1^2}{r}, \\ \frac{\mathrm{d}^2T_1}{\mathrm{d}r^2} &= \frac{\kappa_1\lambda_1^2}{\lambda_2 r}(T_1 - T_2), \\ \text{and} \qquad \frac{\mathrm{d}^2\kappa_2}{\mathrm{d}r^2} &= -\frac{\kappa_1\lambda_1^2}{\lambda_2 r}(\kappa_1 - \kappa_2). \end{aligned}\right\} \tag{8·6}$$

$\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ can be obtained from the expression (6·3) for $\mathrm{d}^2T_1/\mathrm{d}r^2$ and the first two of equations (8·6). $\mathrm{d}^2T_2/\mathrm{d}r^2$ can then be found from the expression for it analogous with (6·3) and these expressions for $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$.

Introducing the expressions (8·3) into (6·4), we obtain

$$\frac{\mathrm{d}^2\kappa_1}{\mathrm{d}r^2} = -\frac{1}{T_1}\left(\kappa_1\frac{\mathrm{d}^2T_1}{\mathrm{d}r^2} + T_2\frac{\mathrm{d}^2\kappa_2}{\mathrm{d}r^2} + \kappa_2\frac{\mathrm{d}^2T_2}{\mathrm{d}r^2}\right), \tag{8·7}$$

which together with the expressions for $\mathrm{d}^2T_1/\mathrm{d}r^2$, $\mathrm{d}^2\kappa_2/\mathrm{d}r^2$ and $\mathrm{d}^2T_2/\mathrm{d}r^2$, found in the manner described, yields an expression for $\mathrm{d}^2\kappa_1/\mathrm{d}r^2$.

If the stored-energy function W has the Mooney form (7·20), then equation (8·4) yields

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}r} = \frac{\lambda_1[C_1(3\lambda_3^2-\lambda_1^2)+C_2\lambda_2^2(\lambda_1^2+\lambda_3^2)]}{r(\lambda_1^2+3\lambda_3^2)(C_1+\lambda_2^2C_2)} \tag{8·8}$$

9. The numerical solution of the problem

In the previous sections, relations have been derived by which the values of λ_1, λ_2, κ_1 and κ_2 at any point of the inflated sheet, initially at $r+\Delta r$, can be calculated if their values are known at a point initially at r. This calculation may be carried out, in principle, for any known form of the stored-energy function. It is, however, very much simplified if the stored-energy function W takes the Mooney form given by equation (5·11). The steps in the calculation may be summarized for this case, λ_1, λ_2, κ_1 and κ_2 being considered known for a given value of r.

Employing the notation $T_1' = T_1/2hC_1$, $T_2' = T_2/2hC_1$ and $\Gamma = C_2/C_1$, we obtain:

(1) $\lambda_3 = 1/\lambda_1\lambda_2$;

(2) $T_1' = \lambda_3(\lambda_1^2-\lambda_3^2)(1+\lambda_2^2\Gamma)$, from (5·12);

(3) $T_2' = \lambda_3(\lambda_2^2-\lambda_3^2)(1+\lambda_1^2\Gamma)$, from (5·12);

(4) $\rho = \lambda_2 r$, from (4·6);

(5) $\mathrm{d}\rho/\mathrm{d}r = \lambda_1(1-\kappa_2^2\rho^2)^{\frac{1}{2}}$, from (5·5);

(6) $\dfrac{\mathrm{d}\lambda_2}{\mathrm{d}r} = \dfrac{1}{r}\left(\dfrac{\mathrm{d}\rho}{\mathrm{d}r} - \lambda_2\right)$, from (4·6);

(7) $$\frac{d\kappa_2}{dr} = \frac{1}{\rho}\frac{d\rho}{dr}(\kappa_1 - \kappa_2),$$ from (5·3);

(8) $$\frac{dT_1'}{dr} = -\frac{1}{\rho}\frac{d\rho}{dr}(T_1' - T_2'),$$ from (5·3);

(9) $$\frac{d\lambda_1}{dr} = \frac{\dfrac{dT_1'}{dr} - [(3\lambda_3^2 - \lambda_1^2) + (\lambda_1^2 + \lambda_3^2)\lambda_2^2\Gamma]\dfrac{\lambda_3}{\lambda_2}\dfrac{d\lambda_2}{dr}}{(\lambda_1^2 + 3\lambda_3^2)(1 + \lambda_2^2\Gamma)\dfrac{\lambda_3}{\lambda_1}},$$ from (5·13);

(10) $$\frac{dT_2'}{dr} = [(\lambda_2^2 + 3\lambda_3^2)(1 + \lambda_1^2\Gamma)]\frac{\lambda_3}{\lambda_2}\frac{d\lambda_2}{dr} + [(3\lambda_3^2 - \lambda_2^2) + (\lambda_2^2 + \lambda_3^2)\lambda_1^2\Gamma]\frac{\lambda_3}{\lambda_1}\frac{d\lambda_1}{dr},$$ from (5·13);

(11) $$\frac{d\kappa_1}{dr} = -\frac{1}{T_1'}\left[\kappa_1\frac{dT_1'}{dr} + \kappa_2\frac{dT_2'}{dr} + T_2'\frac{d\kappa_2}{dr}\right],$$ from (5·10);

(12) $$\frac{d^2\rho}{dr^2} = \frac{1}{\lambda_1}\left(\frac{d\rho}{dr}\frac{d\lambda_1}{dr} - \kappa_1\kappa_2\rho\lambda_1^3\right),$$ from (5·2);

(13) $$\frac{d^2\lambda_2}{dr^2} = \frac{1}{r}\left(\frac{d^2\rho}{dr^2} - 2\frac{d\lambda_2}{dr}\right),$$ from (6·2);

(14) $$\frac{d^2\kappa_2}{dr^2} = \frac{1}{\rho}\left[(\kappa_1 - \kappa_2)\frac{d^2\rho}{dr^2} + \left(\frac{d\kappa_1}{dr} - 2\frac{d\kappa_2}{dr}\right)\frac{d\rho}{dr}\right],$$ from (6·2);

(15) $$\frac{d^2T_1'}{dr^2} = -\frac{1}{\rho}\left[(T_1' - T_2')\frac{d^2\rho}{dr^2} + \left(2\frac{dT_1'}{dr} - \frac{dT_2'}{dr}\right)\frac{d\rho}{dr}\right],$$ from (6·2);

(16) $$\frac{d^2\lambda_1}{dr^2} = \frac{\lambda_1}{\lambda_3}\frac{\dfrac{d^2T_1'}{dr^2} - \Phi}{(\lambda_1^2 + 3\lambda_3^2)(1 + \lambda_2^2\Gamma)},$$

where $$\Phi = [(3\lambda_3^2 - \lambda_1^2) + (\lambda_1^2 + \lambda_3^2)\lambda_2^2\Gamma]\frac{\lambda_3}{\lambda_2}\frac{d^2\lambda_2}{dr^2} - 12(1 + \lambda_2^2\Gamma)\frac{\lambda_3^3}{\lambda_1^2}\left(\frac{d\lambda_1}{dr}\right)^2$$
$$+ 2[-(\lambda_1^2 + 9\lambda_3^2) + \lambda_2^2(\lambda_1^2 - 3\lambda_3^2)\Gamma]\lambda_3^2\frac{d\lambda_1}{dr}\frac{d\lambda_2}{dr}$$
$$+ 2\lambda_3^3[\lambda_1^2(\lambda_1^2 - 6\lambda_3^2) - \Gamma]\left(\frac{d\lambda_2}{dr}\right)^2,$$ from (6·5);

(17) $$\frac{d^2T_2'}{dr^2} = [(\lambda_2^2 + 3\lambda_3^2)(1 + \lambda_1^2\Gamma)]\frac{\lambda_3}{\lambda_2}\frac{d^2\lambda_2}{dr^2}$$
$$+ [(3\lambda_3^2 - \lambda_2^2) + (\lambda_2^2 + \lambda_3^2)\lambda_1^2\Gamma]\frac{\lambda_3}{\lambda_1}\frac{d^2\lambda_1}{dr^2} - 12\lambda_3^2(1 + \lambda_1^2\Gamma)\frac{\lambda_3}{\lambda_2^2}\left(\frac{d\lambda_2}{dr}\right)^2$$
$$+ 2[-(\lambda_2^2 + 9\lambda_3^2) + \lambda_1^2(\lambda_2^2 - 3\lambda_3^2)\Gamma]\lambda_3^2\frac{d\lambda_1}{dr}\frac{d\lambda_2}{dr}$$
$$+ 2\lambda_3^3[\lambda_2^2(\lambda_2^2 - 6\lambda_3^2) - \Gamma]\left(\frac{d\lambda_1}{dr}\right)^2,$$ from (6·6);

(18) $$\frac{d^2\kappa_1}{dr^2} = -\frac{1}{T_1'}\left[2\frac{d\kappa_1}{dr}\frac{dT_1'}{dr} + \kappa_1\frac{d^2T_1'}{dr^2} + T_2'\frac{d^2\kappa_2}{dr^2} + 2\frac{d\kappa_2}{dr}\frac{dT_2'}{dr} + \kappa_2\frac{d^2T_2'}{dr^2}\right],$$ from (6·4).

By means of this scheme, the values of the first and second derivatives of λ_1, λ_2, κ_1 and κ_2 can be calculated at the point r. Then, their values at the point $(r+\Delta r)$ can be calculated approximately by formulae of the type

$$[\lambda_1]_{r+\Delta r} = [\lambda_1]_r + \left[\frac{d\lambda_1}{dr}\right]_r \Delta r + \frac{1}{2}\left[\frac{d^2\lambda_1}{dr^2}\right]_r (\Delta r)^2. \qquad (9\cdot1)$$

The values so obtained for $[\lambda_1]_{r+\Delta r}$, etc., can be made as accurate as desired by making Δr sufficiently small.

If the values of λ $(=\lambda_1=\lambda_2)$ and κ $(=\kappa_1=\kappa_2)$ at the pole of the inflated sheet (where $r=0$) are known or assumed, then by successive application of the formulae, the values of λ_1, λ_2, κ_1 and κ_2 can be calculated for all values of r. This calculation has been carried out for values of λ of 1·5, 3·0, 4·5 and 6·0 employing values of the ratio Γ in the Mooney form for the stored-energy function of 0 and 0·1. The values of λ_1 and λ_2 obtained are given in tables 1 and 2. κ may be given any value in arbitrary units without any loss in generality in the solution. ρ and r are, of course, measured in units corresponding to those of κ. In the calculations the following values were given to κ for reasons of arithmetical convenience: $\kappa = 0\cdot1$ when $\lambda = 4\cdot5$ or $6\cdot0$; $\kappa = 0\cdot2$ when $\lambda = 3\cdot0$; and $\kappa = 0\cdot3$ when $\lambda = 1\cdot5$.

Table 1. Calculated values of λ_1 and λ_2 for $\Gamma=0$

	$\lambda=1\cdot5,\ \kappa=0\cdot3$		$\lambda=3\cdot0,\ \kappa=0\cdot2$		$\lambda=4\cdot5,\ \kappa=0\cdot1$		$\lambda=6\cdot0,\ \kappa=0\cdot1$	
r	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
0	1·5000	1·5000	3·0000	3·0000	4·5000	4·5000	6·0000	6·0000
0·4	1·4918	1·4892	2·9577	2·9575	4·4639	4·4639	5·9149	5·9149
0·8	1·4682	1·4574	2·8377	2·8369	4·3590	4·3589	5·6732	5·6732
1·2	1·4327	1·4065	2·6577	2·6554	4·1942	4·1941	5·3112	5·3112
1·6	1·3906	1·3391	2·4415	2·4373	3·9829	3·9825	4·8755	4·8753
2·0	1·3482	1·2578	2·2143	2·2031	3·7406	3·7401	4·4106	4·4102
2·4	1·3127	1·1643	1·9954	1·9726	3·4820	3·4810	3·9507	3·9500
2·8	1·2916	1·0588	1·7992	1·7545	3·2203	3·2185	3·5181	3·5166
3·0026	1·2893	1·0000	—	—	—	—	—	—
3·2			1·6365	1·5542	2·9641	2·9610	3·1248	3·1218
3·6			1·5148	1·3713	2·7202	2·7149	2·7756	2·7699
4·0			1·4367	1·2048	2·4934	2·4845	2·4717	2·4601
4·4			1·4008	1·0509	2·2861	2·2714	2·2095	2·1891
4·5376			1·3977	1·0000	—	—	—	—
4·8					2·0996	2·0760	1·9884	1·9528
5·2					1·9353	1·8979	1·8067	1·7468
5·6					1·7939	1·7360	1·6634	1·5659
6·0					1·6759	1·5890	1·5568	1·4060
6·4					1·5816	1·4551	1·4844	1·2636
6·8					1·5102	1·3326	1·4422	1·1347
7·2					1·4604	1·2196	1·4261	1·0163
7·2576					—	—	1·4258	1·0000
7·6					1·4303	1·1145		
8·0					1·4181	1·0157		
8·0653					1·4176	1·0000		

The solutions to a number of different problems can be obtained from tables 1 and 2.

For example, suppose the diaphragm to be clamped in its undeformed state on a circle at $r = 1$ cm (say). Then, when $r = 1$ cm, $\lambda_2 = 1$. We must choose the units of r, ρ and κ in the tables in such a way that this boundary condition is satisfied. For example, in the case when $\lambda = 6\cdot0$ and $\Gamma = 0$, we see from table 1 that $\lambda_2 = 1$ when $r = 7\cdot2576$ units. Thus, we have to choose, as the unit for r, 0·13779 (= 1/7·2576) cm. In a similar way we can determine in cm the value of the unit which is appropriate to each of the other cases. Having so determined the value of the unit in each case, we can obtain from the tables the dependence of λ_1 and λ_2 on r appropriate to the problem under consideration. These results are plotted in figures 3 and 4.

Table 2. Calculated values of λ_1 and λ_2 for $\Gamma = 0{\cdot}1$

	$\lambda = 1{\cdot}5,\ \kappa = 0{\cdot}3$		$\lambda = 3{\cdot}0,\ \kappa = 0{\cdot}2$		$\lambda = 4{\cdot}5,\ \kappa = 0{\cdot}1$		$\lambda = 6{\cdot}0,\ \kappa = 0{\cdot}1$	
r	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
0	1·5000	1·5000	3·0000	3·0000	4·5000	4·5000	6·0000	6·0000
0·4	1·4941	1·4899	2·9875	2·9671	4·5001	4·4757	6·0151	5·9473
0·8	1·4770	1·4601	2·9494	2·8693	4·5005	4·4028	6·0621	5·7878
1·2	1·4504	1·4116	2·8836	2·7092	4·5008	4·2806	6·1454	5·5168
1·6	1·4173	1·3456	2·7868	2·4922	4·5005	4·1085	6·2745	5·1263
2·0	1·3822	1·2639	2·6566	2·2278	4·4985	3·8857	6·4636	4·6046
2·4	1·3514	1·1675	2·4938	1·9302	4·4931	3·6116	6·7336	3·9347
2·8	1·3330	1·0566	2·3077	1·6186	4·4810	3·2865	7·0863	3·1170
2·9845	1·3318	1·0000	—	—	—	—	—	—
3·2			2·1211	1·3112	4·4567	2·9126	7·4587	2·1644
3·6			1·9740	1·0274	4·4089	2·5029	7·5452	1·1749
3·6409			1·9635	1·0000	—	—	—	—
3·6757					—	—	7·4795	1·0000
4·0					4·3278	2·0678		
4·4					4·2019	1·6262		
4·8					4·0300	1·2077		
5·0150					3·9244	1·0000		

In figures 5 and 6, κ_1 and κ_2 are plotted against r and in figures 7 and 8 T_1' and T_2' are plotted against r. It will be noticed from figures 5 and 6 that the variations of κ_2 throughout the sheet are less for the neo-Hookean material, for which $\Gamma = 0$, than for the material for which $\Gamma = 0{\cdot}1$, i.e. the shape of the deformed sheet is more nearly spherical in the former than in the latter case, particularly for high degrees of inflation.

From figures 7 and 8 we observe that whereas T_2' falls steadily as we progress outwards from the pole, T_1' falls slightly between the pole and the equator and then commences to rise again near the edge of the sheet, the rise being more pronounced for the higher degrees of inflation. The assumption made by Flint & Naunton (1937) that T_1' is approximately constant throughout the sheet applies fairly well over quite a large area, particularly for materials for which the neo-Hookean form of stored-energy function is appropriate.

We shall suppose that the position of a point on the inflated diaphragm, which is at a radial distance r from its centre in its undeformed state, is specified by its radial distance ρ from the axis of symmetry and its distance ζ from the pole of the inflated sheet, measured parallel to this axis. Then, the shape of the deformed body may be described by a relation between ζ and ρ. Since

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}r}\right)^2 = \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right)^2, \tag{9·2}$$

from (5·5) and the first of equations (4·6), we obtain

$$\frac{\mathrm{d}\zeta}{\mathrm{d}r} = \kappa_2 \lambda_1 \rho. \tag{9·3}$$

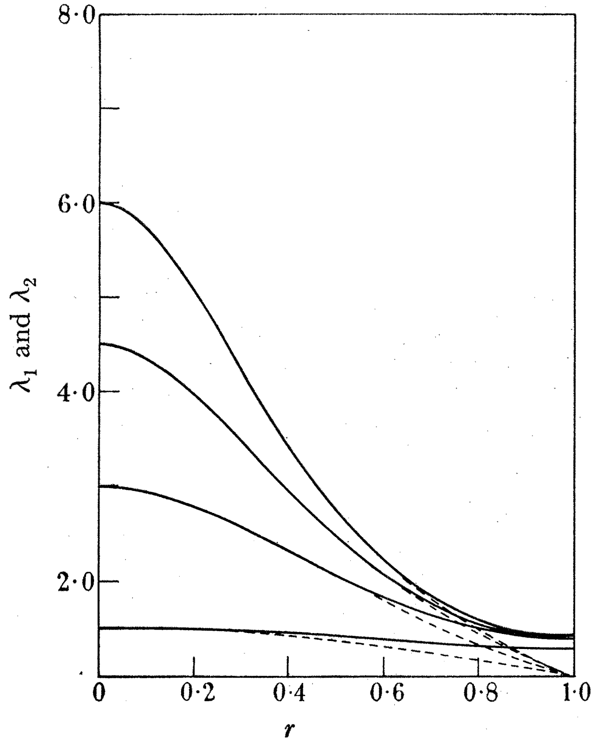

FIGURE 3. Calculated $\lambda_1 - r$ and $\lambda_2 - r$ curves for $\Gamma = 0$. Full lines give $\lambda_1 - r$ and broken lines $\lambda_2 - r$.

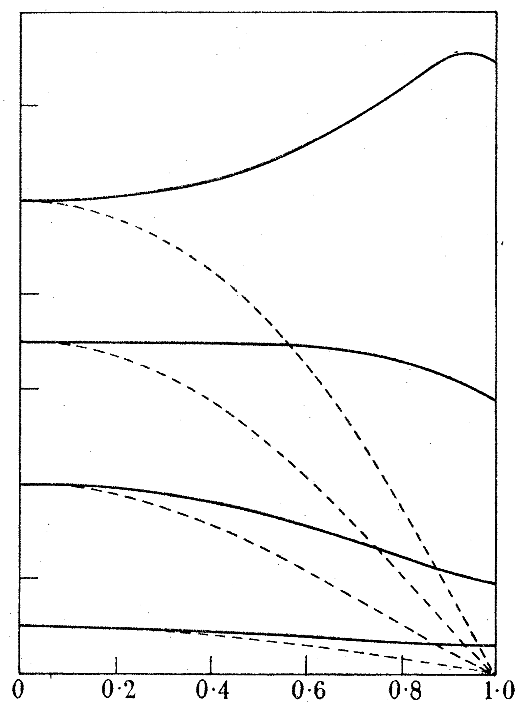

FIGURE 4. Calculated $\lambda_1 - r$ and $\lambda_2 - r$ curves for $\Gamma = 0{\cdot}1$. Full lines give $\lambda_1 - r$ and broken lines $\lambda_2 - r$.

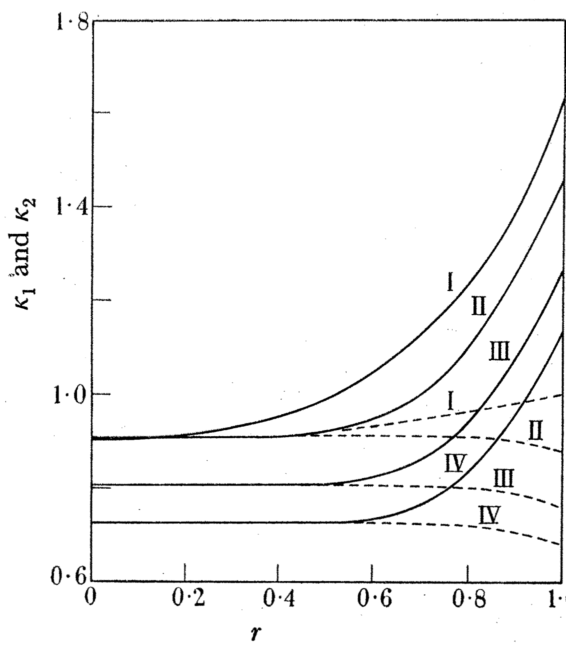

FIGURE 5. Calculated $\kappa_1 - r$ and $\kappa_2 - r$ curves for $\Gamma = 0$. Full lines give $\kappa_1 - r$ and broken lines $\kappa_2 - r$. I, $\lambda = 1{\cdot}5$; II, $\lambda = 3{\cdot}0$; III, $\lambda = 4{\cdot}5$; IV, $\lambda = 6{\cdot}0$.

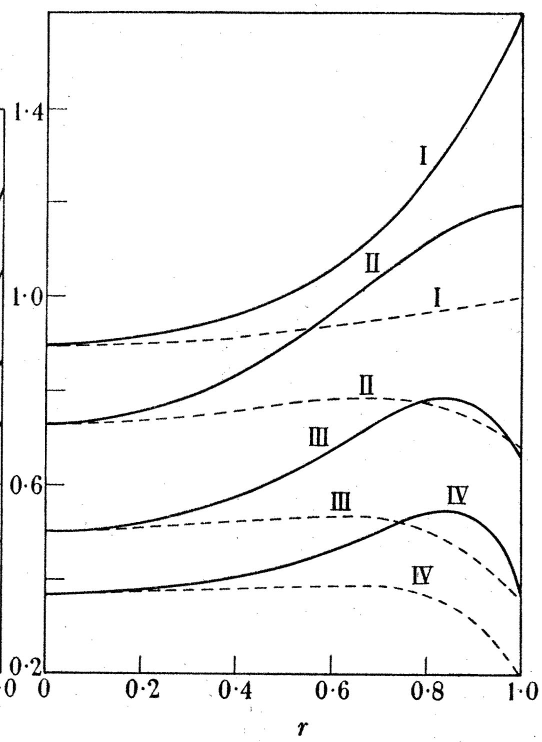

FIGURE 6. Calculated $\kappa_1 - r$ and $\kappa_2 - r$ curves for $\Gamma = 0{\cdot}1$. Full lines give $\kappa_1 - r$ and broken lines $\kappa_2 - r$. I, $\lambda = 1{\cdot}5$; II, $\lambda = 3{\cdot}0$; III, $\lambda = 4{\cdot}5$; IV, $\lambda = 6{\cdot}0$.

Differentiating this expression, we obtain

$$\frac{d^2\zeta}{dr^2} = \lambda_1\rho\frac{d\kappa_2}{dr} + \kappa_2\lambda_1\frac{d\rho}{dr} + \kappa_2\rho\frac{d\lambda_1}{dr}. \tag{9·4}$$

Employing the relations (9·3) and (9·4), $d\zeta/dr$ and $d^2\zeta/dr^2$ can be calculated at each point of the deformed body, and by means of the relation

$$[\zeta]_{r+\Delta r} = [\zeta]_r + \left[\frac{d\zeta}{dr}\right]_r \Delta r + \frac{1}{2}\left[\frac{d^2\zeta}{dr^2}\right]_r (\Delta r)^2,$$

ζ can be calculated at successive points, bearing in mind that $\zeta = 0$ when $r = 0$.

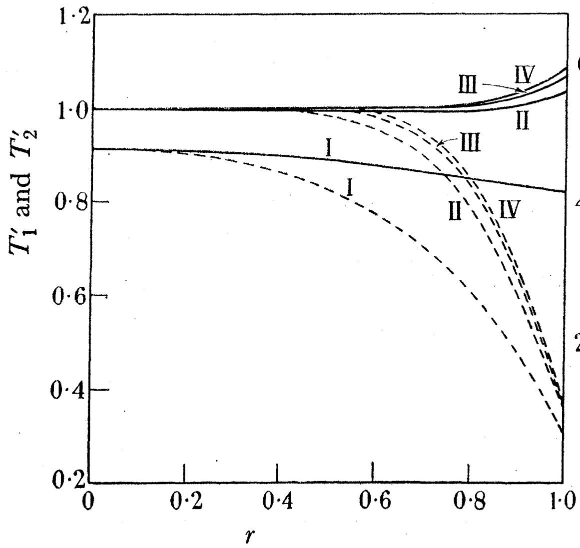

FIGURE 7. Calculated $T_1'-r$ and $T_2'-r$ curves for $\Gamma = 0$. Full lines give $T_1'-r$ and broken lines $T_2'-r$. I, $\lambda = 1·5$; II, $\lambda = 3·0$; III, $\lambda = 4·5$; IV, $\lambda = 6·0$.

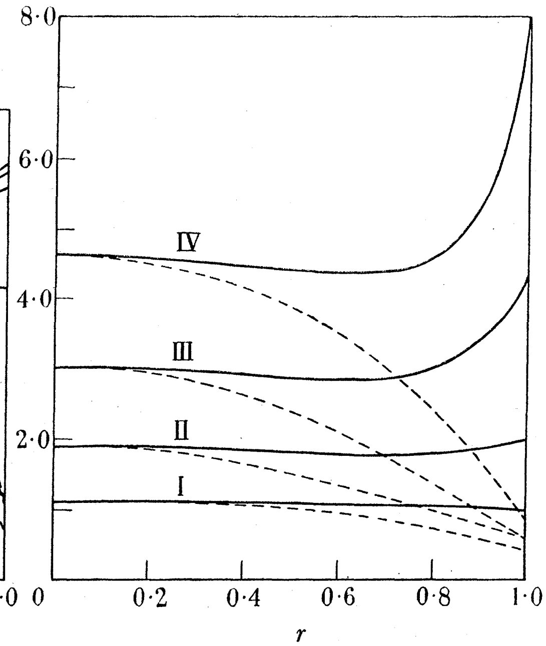

FIGURE 8. Calculated $T_1'-r$ and $T_2'-r$ curves for $\Gamma = 0·1$. Full lines give $T_1'-r$ and broken lines $T_2'-r$. I, $\lambda = 1·5$; II, $\lambda = 3·0$; III, $\lambda = 4·5$; IV, $\lambda = 6·0$.

In this manner, the forms assumed by the deformed body have been calculated on the assumption that the stored-energy function for the material has the Mooney form with $\Gamma = 0$ and 0·1 and for extension ratios λ at the pole of magnitude 1·5, 3·0, 4·5 and 6·0. The axial cross-sections obtained are shown in figures 9 and 10. The axial cross-sections obtained experimentally by Treloar (1944) are also shown in figure 10 for purposes of comparison.

In carrying out the calculations, the values of Δr must, of course, be chosen so that adequate accuracy is obtained without excessively increasing the labour involved in the computation. Provided that the computation is carried out to a sufficient number of significant figures, the errors introduced in each stage of the numerical integration arise from the neglect in equation (9·1) of terms of higher degree than the second in Δr and from inaccuracies in the values of $[d\lambda_1/dr]_r$ and $[d^2\lambda_1/dr^2]_r$ arising from such neglect at earlier stages of the integration. It is thus seen that the error introduced at each stage of the integration will tend to increase as the integration proceeds, due to the cumulative effect of the errors introduced at the earlier stages (i.e. near the pole). It is therefore desirable to employ smaller values of Δr near the pole than are appropriate at later stages of the calculation. With the values of κ employed, it was found desirable to employ a value of $\Delta r = 0{\cdot}1$ near the pole, while values of two and three times this amount were used at the higher value of r.

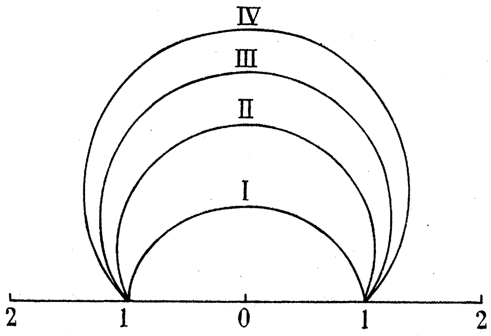

FIGURE 9. Profiles of inflated sheet calculated for $\Gamma = 0$. I, $\lambda = 1{\cdot}5$; II, $\lambda = 3{\cdot}0$; III, $\lambda = 4{\cdot}5$; IV, $\lambda = 6{\cdot}0$.

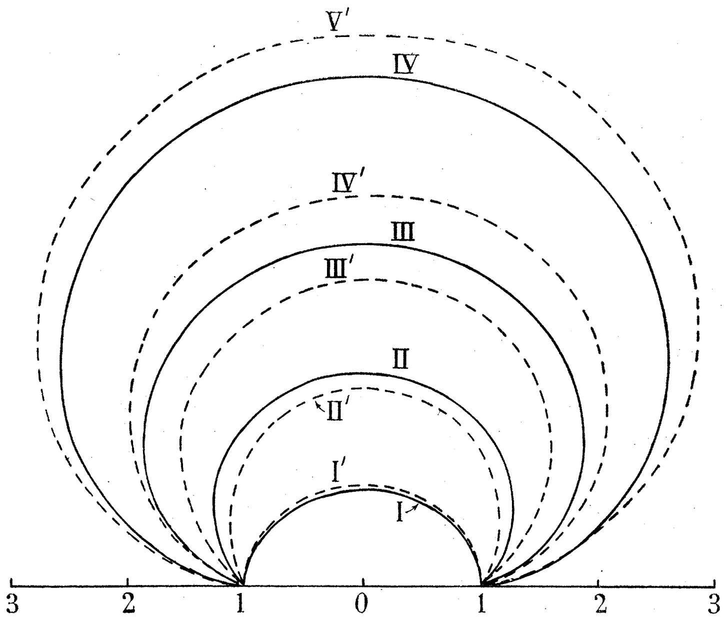

FIGURE 10. Profiles of inflated sheet calculated for $\Gamma = 0{\cdot}1$ (full lines) and obtained by Treloar experimentally (broken lines). I, $\lambda = 1{\cdot}5$; II, $\lambda = 3{\cdot}0$; III, $\lambda = 4{\cdot}5$; IV, $\lambda = 6{\cdot}0$; I′, $\lambda = 1{\cdot}49$; II′, $\lambda = 3{\cdot}36$; III′, $\lambda = 4{\cdot}7$; IV′, $\lambda = 5{\cdot}35$; V′, $\lambda = 5{\cdot}9$.

Some estimate of the magnitude of the error introduced at any stage of the numerical integration, by the neglect in the expressions of the type (9·1) of terms of higher degree than the second in Δr, may be obtained in the following manner. Suppose, for example, we are concerned with the error introduced in calculating $[\lambda_1]_{r+\Delta r}$ from the solution at the point r. We calculate the value of $[\lambda_1]_{r+2\Delta r}$ in a single stage of numerical integration using the interval $2\Delta r$ and compare it with the value obtained by two stages of numerical integration using intervals Δr. It was found that the difference between the values of $[\lambda_1]_{r+2\Delta r} - [\lambda_1]_r$

calculated by these two methods were always less than 0·1 %. Since the errors introduced in each stage of the integration tend to zero as Δr decreases, it may be concluded that the errors introduced in each stage of the integration by the neglect, in that stage, of terms of higher degree in Δr than the second in equation (9·1) are less than 0·1 % of $[\lambda_1]_{r+\Delta r}-[\lambda_1]_r$. Similar considerations were found to apply to λ_2, κ_1 and κ_2. If the errors introduced in each stage of the calculation had no effect on succeeding stages then the calculated values of $\lambda_1-\lambda$, $\lambda_2-\lambda$, $\kappa_1-\kappa$ and $\kappa_2-\kappa$ would be in error by about 0·1 % over the whole sheet. Owing to the cumulative effect of the errors, the resultant errors in these quantities will be considerably greater, particularly for the larger values of r, but it is estimated that they should not exceed 2 %.

10. Comparison of theoretical and experimental results

It will be observed from a comparison of figures 3 and 4 that the deformation obtained is very sensitive to variations in the stored-energy function W and that the shape of the λ_1-r curves are more dependent upon the value of Γ than are those of the λ_2-r curves.

Treloar (1944) has measured the values of λ_1 and λ_2 over the surface of an inflated diaphragm of natural rubber for various degrees of inflation. The values he obtained for λ_1 and λ_2 are plotted against r in figure 11 and may be compared directly with our calculated values. It is seen that the curves of figure 4, calculated on the assumption of a Mooney form for the stored-energy function, with $\Gamma = 0{\cdot}1$, agree more closely with Treloar's results than do those of figure 3 which assume that the material is neo-Hookean (i.e. $\Gamma = 0$). There are, however, appreciable disparities between the curves of figures 4 and 11. These may be explained to some extent in terms of the known departure of the stored-energy function of a natural rubber vulcanizate from the Mooney form assumed in calculating the curves of figure 4.

Rivlin & Saunders (1951) have found, for a particular pure gum vulcanizate, that the stored-energy function W is such that $\partial W/\partial I_1$ is substantially independent of I_1 and I_2, while $\partial W/\partial I_2$ is independent of I_1 and decreases as I_2 increases. For small deformations, where I_2 is little greater than 3, $(\partial W/\partial I_2)/(\partial W/\partial I_1)$ is about 0·25 and falls steadily as I_2 increases, reaching a value of about 0·05 at $I_2 = 30$. For values of I_2 between about 30 and 120, $(\partial W/\partial I_2)/(\partial W/\partial I_1)$ lies between 0·05 and 0·03.

Although these results were obtained for a particular vulcanizate, there are strong indications that the form of W is not very sensitive to variations in the vulcanizate, so that it is to be expected that the general character of the variation of W with I_1 and I_2 obtained from the experiments of Rivlin & Saunders will be applicable to the vulcanizate used in Treloar's experiment.

For the type of deformation considered, I_1 and I_2 vary considerably over the surface of the sheet in any state of inflation and are highly dependent on the state of inflation. This can be seen from table 3, in which the values of I_1 and I_2 are given for various values of r. These are obtained from equations (4·1) and (4·2) using the values of λ_1 and λ_2 obtained by Treloar experimentally.

Comparing the curves for which $\lambda = 1{\cdot}5$ (1·49 in the experimental case), we see that the λ_2-r curves calculated for $\Gamma = 0$ and $\Gamma = 0{\cdot}1$ are nearly identical with each other and with the experimental curve. Again, within experimental error, Treloar's $\lambda_1 - r$ curve for $\lambda = 1{\cdot}49$ agrees with the calculated $\lambda_1 - r$ curve for $\lambda = 1{\cdot}50$ and $\Gamma = 0{\cdot}1$. The latter is only very slightly above the corresponding curve for $\Gamma = 0$. The insensitivity of the calculated curves to the value of Γ is in accord with the agreement of the experimental and calculated curves,

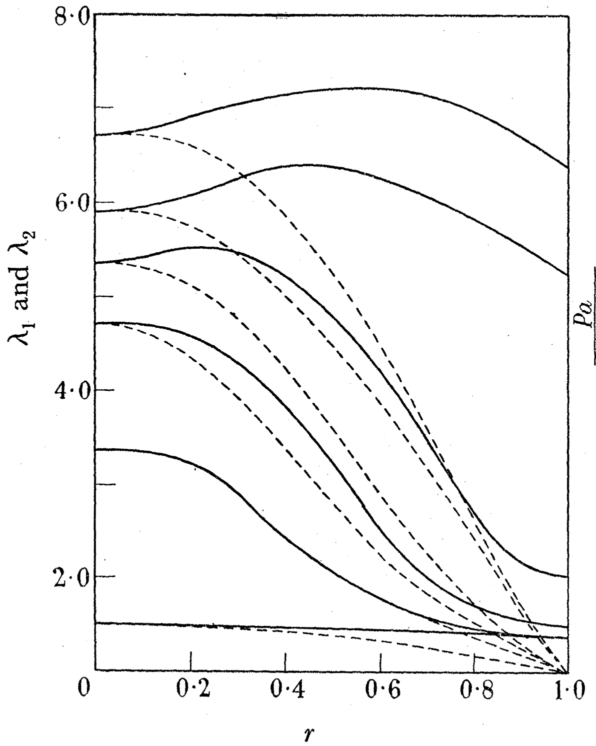

FIGURE 11. $\lambda_1 - r$ and $\lambda_2 - r$ curves obtained experimentally by Treloar. Full lines give $\lambda_1 - r$ and broken lines $\lambda_2 - r$.

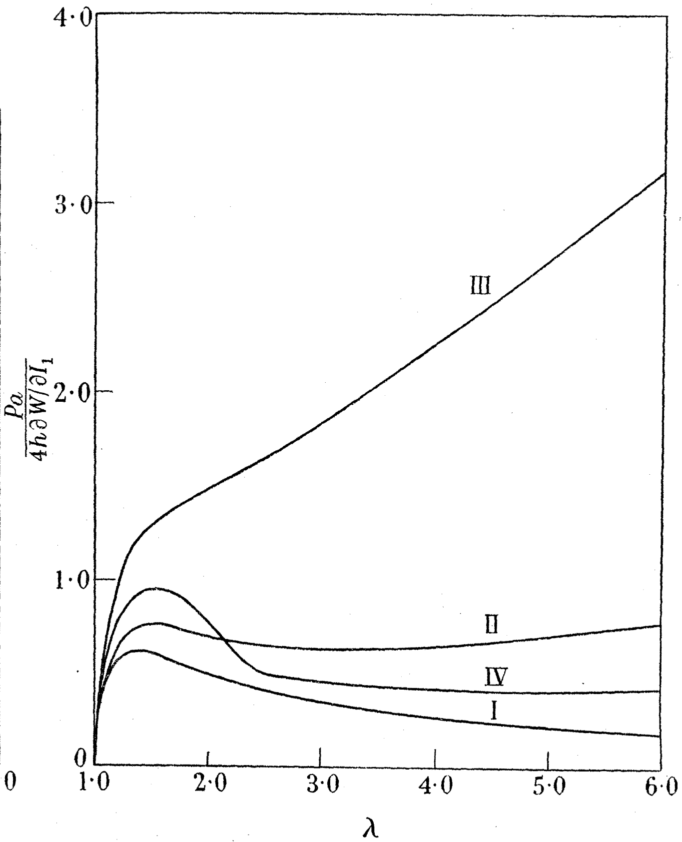

FIGURE 12. Variation of inflating pressure with degree of inflation for spherical balloon. I, $\Gamma = 0$; $\Gamma = 0{\cdot}1$; III, $\Gamma = 0{\cdot}5$; IV, Γ varying in the manner obtained experimentally by Rivlin & Saunders.

TABLE 3. VALUES OF I_1 AND I_2 IN TRELOAR'S EXPERIMENTAL RESULTS

	$\lambda = 1{\cdot}49$		$\lambda = 3{\cdot}36$		$\lambda = 4{\cdot}7$		$\lambda = 5{\cdot}35$		$\lambda = 5{\cdot}9$		$\lambda = 6{\cdot}7$	
r	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2
0	4·64	5·83	22·6	127·6	44·2	488	57·2	819	69·6	1212	89·8	2015
0·1	4·61	5·79	22·5	126·1	43·2	465	57·1	816	69·7	1216	90·2	2033
0·2	4·53	5·60	20·8	107·7	39·7	393	56·3	790	69·7	1210	91·8	2106
0·3	4·45	5·43	16·8	70·9	33·8	284	52·6	682	68·9	1164	89·2	1969
0·4	4·38	5·24	13·1	36·9	26·3	170	45·2	490	65·9	1025	85·4	1759
0·5	4·26	5·00	8·9	19·9	18·5	84	36·0	299	61·1	827	79·5	1436
0·6	4·11	4·70	6·5	10·9	11·4	32	26·3	149	54·4	597	72·2	1053
0·7	3·94	4·36	5·1	7·0	7·4	14	17·4	63	47·2	383	64·2	670
0·8	3·74	3·99	4·2	4·9	5·4	7·5	10·4	23	39·9	201	55·6	337
0·9	3·56	3·68	3·7	3·9	4·2	4·7	6·8	9·2	33·4	81	48·1	138
1·0	3·37	3·37	3·5	3·5	3·7	3·7	5·4	5·4	28·6	28·6	41·6	41·6

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

in spite of the variation of I_2 and the consequent variation of $(\partial W/\partial I_2)/(\partial W/\partial I_1)$ over the deformed sheet.

For the higher values of λ, it is not possible to discuss the disparities between the experimental and calculated curves in this manner owing to the absence of adequate information on the manner in which $\partial W/\partial I_1$ and $\partial W/\partial I_2$ behave for the relevant values of I_1 and I_2. However, it is possible to discuss the general behaviour of the experimental curves in the neighbourhood of $r = 0$ in terms of the formulae for $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ at $r = 0$ obtained in § 7 even in the absence of such detailed information.

For $\lambda = 1{\cdot}49$, we have at $r = 0$, from (7·15),

$$\left.\begin{aligned}\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2} &= \frac{-\frac{1}{4}\kappa^2\lambda^3\left[1{\cdot}73\dfrac{\partial W}{\partial I_1} - 2{\cdot}22\dfrac{\partial W}{\partial I_2} - 3{\cdot}66A\right]}{1{\cdot}27\left(\dfrac{\partial W}{\partial I_1} + 2{\cdot}22\dfrac{\partial W}{\partial I_2}\right) + 3{\cdot}66A},\\ \frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2} &= \frac{-\frac{1}{4}\kappa^2\lambda^3\left[2{\cdot}27\dfrac{\partial W}{\partial I_1} + 3{\cdot}03\dfrac{\partial W}{\partial I_2} + 3{\cdot}66A\right]}{1{\cdot}27\left(\dfrac{\partial W}{\partial I_1} + 2{\cdot}22\dfrac{\partial W}{\partial I_2}\right) + 3{\cdot}66A},\end{aligned}\right\} \qquad (10{\cdot}1)$$

where $$A = \frac{\partial^2 W}{\partial I_1^2} + 4{\cdot}44\frac{\partial^2 W}{\partial I_1 \partial I_2} + 4{\cdot}93\frac{\partial^2 W}{\partial I_2^2}.$$

At $r = 0$, $I_2 = 5{\cdot}83$ (from table 3). From the results of Rivlin & Saunders,

$$\partial^2 W/\partial I_1^2 = \partial^2 W/\partial I_1 \partial I_2 = 0 \quad \text{and} \quad (\partial W/\partial I_2)/(\partial W/\partial I_1) \approx 0{\cdot}13$$

at $I_2 = 5{\cdot}83$, while $(\partial^2 W/\partial I_2^2)/(\partial W/\partial I_1) \approx -4\times 10^{-3}$. We see that in this case the contribution of A to the values of $\mathrm{d}^2\lambda_1/\mathrm{d}r^2$ and $\mathrm{d}^2\lambda_2/\mathrm{d}r^2$ will be small and equations (10·1) yield approximately

$$\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2} = -0{\cdot}24\kappa^2\lambda^3 \quad \text{and} \quad \frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2} = -0{\cdot}41\kappa^2\lambda^3.$$

The $\lambda_1 - r$ and $\lambda_2 - r$ curves should both be concave to the r-axis at $r = 0$, λ_2 falling approximately twice as fast as λ_1, as r increases from 0. It is seen from figure 11 that this is in agreement with the experimental result.

For $\lambda = 3{\cdot}36$, we have, at $r = 0$,

$$\left.\begin{aligned}\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}r^2} &= \frac{-\frac{1}{4}\kappa^2\lambda^3\left[2\dfrac{\partial W}{\partial I_1} - 11{\cdot}3\dfrac{\partial W}{\partial I_2} - 22{\cdot}6A\right]}{\dfrac{\partial W}{\partial I_1} + 11{\cdot}3\dfrac{\partial W}{\partial I_2} + 22{\cdot}6A},\\ \frac{\mathrm{d}^2\lambda_2}{\mathrm{d}r^2} &= \frac{-\frac{1}{4}\kappa^2\lambda^3\left[2\dfrac{\partial W}{\partial I_1} + 11{\cdot}3\dfrac{\partial W}{\partial I_2} + 22{\cdot}6A\right]}{\dfrac{\partial W}{\partial I_1} + 11{\cdot}3\dfrac{\partial W}{\partial I_2} + 22{\cdot}6A},\end{aligned}\right\} \qquad (10{\cdot}2)$$

where $$A = \frac{\partial^2 W}{\partial I_1^2} + 22{\cdot}6\frac{\partial^2 W}{\partial I_1 \partial I_2} + 128\frac{\partial^2 W}{\partial I_2^2}.$$

At $r=0$, $I_2 = 127{\cdot}6$ (from table 3). Taking $(\partial W/\partial I_2)/(\partial W/\partial I_1)$ to have the value 0·035 found by Rivlin & Saunders, equations (10·2) yield

$$\left.\begin{aligned}\frac{d^2\lambda_1}{dr^2} &= \frac{-\frac{1}{4}\kappa^2\lambda^3[2-0{\cdot}39-22{\cdot}6A']}{1{\cdot}39+22{\cdot}6A'}\\ \text{and}\qquad \frac{d^2\lambda_2}{dr^2} &= \frac{-\frac{1}{4}\kappa^2\lambda^3[2+0{\cdot}39+22{\cdot}6A']}{1{\cdot}39+22{\cdot}6A'},\end{aligned}\right\} \qquad (10{\cdot}3)$$

where $$A' = A/(\partial W/I_1).$$

The experimental results for this value of λ, indicate that $d^2\lambda_1/dr^2$ and $d^2\lambda_2/dr^2$ are both negative at $r=0$ and approximately equal. In order to obtain agreement with these results, we must have for the rubber used in the experiments $A' \approx -0{\cdot}017$. If we take the non-zero value of A' to be due predominantly to the non-zero value of $\partial^2 W/\partial I_2^2$, we find that

$$(\partial^2 W/\partial I_2^2)/(\partial W/\partial I_1) \approx -1{\cdot}35\times 10^{-4}$$

for the particular rubber employed. It is clear that the precise relationship between $d^2\lambda_1/dr^2$ and $d^2\lambda_2/dr^2$ for this value of λ depends very critically on the relation between $\partial W/\partial I_2$ and A.

It has been pointed out in § 7 that so long as $\partial^2 W/\partial I_2^2$ or $\partial^2 W/\partial I_1\,\partial I_2$ are not zero, they may be the controlling terms in determining the values of $d^2\lambda_1/dr^2$ and $d^2\lambda_2/dr^2$ at $r=0$, given by equations (7·15), for sufficiently large values of λ on account of the large values of the factors by which they are multiplied. We see, for example, that provided $\partial^2 W/\partial I_2^2$ is much greater than $\frac{1}{2}\lambda^{-4}\,\partial W/\partial I_2$ and $\lambda^{-6}\,\partial W/\partial I_1$, $d^2\lambda_1/dr^2$ and $d^2\lambda_2/dr^2$ are given by

$$\frac{d^2\lambda_1}{dr^2} = -\frac{d^2\lambda_2}{dr^2} = \tfrac{1}{4}\kappa^2\lambda^3.$$

Now, in the experimental curves for which $\lambda = 5{\cdot}9$, the factors $\frac{1}{2}\lambda^{-4}$ and λ^{-6} are about 4×10^{-4} and $2{\cdot}4\times 10^{-5}$ respectively, so even exceedingly small values of $\partial^2 W/\partial I_2^2$ would make A the controlling term in equations (7·15). This is the case *a fortiori* for the curves for which $\lambda \doteq 6{\cdot}7$, and it is therefore hardly surprising that the values of $d^2\lambda_1/dr^2$ and $d^2\lambda_2/dr^2$ obtained experimentally for these values of λ are equal in magnitude and opposite in sign, the $\lambda_1 - r$ curve being convex and the $\lambda_2 - r$ curve concave to the r-axis.

It appears then that the experimental curves for which $\lambda = 1{\cdot}49$ represent a situation in which A has a negligible effect in determining the values of $d^2\lambda_1/dr_2$ and $d^2\lambda_2/dr^2$ at $r=0$. In the case when $\lambda = 3{\cdot}36$, the behaviour is determined by a balance between the values of $\partial W/\partial I_2$ and A while in the cases when $\lambda = 5{\cdot}9$ and 6·7, the terms A in equations (7·15) have become controlling. The curves for which $\lambda = 4{\cdot}7$ and 5·35 presumably represent intermediate cases.

A FURTHER APPLICATION OF THE THEORY

11. Inflation of a spherical shell

The problem of the relation between the pressure necessary to inflate a thick spherical shell of incompressible highly elastic material, isotropic in its undeformed state, and the amount of inflation has been solved by Green & Shield (1950). They have obtained from their results the following formula for the internal pressure P required to inflate a thin shell of the material of initial thickness h and radius a (such that $h \ll a$) to a radius λa:

$$P = \frac{4h}{a\lambda}\left(1-\frac{1}{\lambda^6}\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right), \tag{11·1}$$

where W is the stored-energy function for the material.

This result can, of course, be obtained by application of the equations given in §§ 3 and 4 of this paper, or more directly from the consideration that in the deformed spherical shell each element of the shell is in a state of pure homogeneous deformation, the extension ratios being λ in any direction tangential to the shell and $1/\lambda^2$ in the radial direction.

The normal component of stress t in any direction tangential to the shell is thus given (Rivlin 1948 *b*) by

$$t = 2\left(\lambda^2-\frac{1}{\lambda^4}\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right). \tag{11·2}$$

The thickness of the shell in its deformed state is h/λ^2, so that T, the tension in the shell, per unit element of length on it, measured in the deformed state, is given by

$$T = \frac{2h}{\lambda^2}\left(\lambda^2-\frac{1}{\lambda^4}\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right). \tag{11·3}$$

From this equation and the relation $P = 2T/a\lambda$ we obtain the result (11·1). Green & Shield have pointed out that if the stored-energy function W for the material has the form (5·11), suggested by Mooney, with $C_1 = 0$, then P increases monotonically with λ. On the other hand, if $C_2 = 0$, so that the material is of the incompressible, neo-Hookean type, then P has a maximum when $\lambda = \sqrt[6]{7}$.

Figure 12 shows the relation between $Pa/(4h\,\partial W/\partial I_1)$ and λ obtained from (11·1) if W is considered to vary with I_1 and I_2 in the manner found experimentally by Rivlin & Saunders (1951) for a particular vulcanizate of natural rubber. For comparison the results obtained when the stored-energy function W has the Mooney form with $\Gamma = 0$, 0·1 and 0·5 are also shown.

It is seen that when W has the form determined experimentally by Rivlin & Saunders, the pressure rises with inflation at first, then falls with further inflation, finally rising again slowly at high values of λ. The same type of behaviour obtains if the material has a stored-energy function of the Mooney form with $\Gamma = 0·1$, but the fall in pressure is much less pronounced than in the previous case. This contrasts with the behaviour of the plane diaphragm during inflation where, for $\Gamma = 0·1$, P increases monotonically with λ. This difference in behaviour is to be expected, since the relation $P = 2\kappa T$ shows that for a given value of λ, P is proportional to the curvature of the inflated sheet. In the region in which the fall in pressure occurs for the spherical shell, i.e. $\lambda = 1·5$ to 3·0 approximately, the curvature of the shell is roughly halved, whereas for the plane diaphragm it can be seen from figure 6 that the fall in the value of the curvature at the pole is much smaller.

Since the fall in P during the inflation of a spherical shell is more pronounced in the case of a stored-energy function having the form determined experimentally by Rivlin & Saunders than in that of a Mooney form with $\Gamma = 0·1$, the foregoing results are not inconsistent with the fall in pressure observed experimentally by Rivlin & Saunders (1951) when a rubber diaphragm is inflated.

In principle the value of λ for which P has a stationary value may be obtained analytically by solving the equation $\mathrm{d}P/\mathrm{d}\lambda = 0$. If W has the Mooney form (5·11) we obtain

$$\frac{a}{4h}\frac{\mathrm{d}P}{\mathrm{d}\lambda} = -\left(\frac{1}{\lambda^2}-\frac{7}{\lambda^8}\right)C_1+\left(1+\frac{5}{\lambda^6}\right)C_2, \tag{11·4}$$

which yields, with $\Gamma = C_2/C_1$ and $x = \lambda^2$,

$$\Gamma x^4 - x^3 + 5\Gamma x + 7 = 0. \tag{11·5}$$

By Descartes's rule of signs, we see that if $\Gamma > 0$ equation (11·5) cannot have more than two real positive roots, so that there are at most two positive values of λ for which P has a stationary value. If C_1 and C_2 are both positive it is evident from equation (11·4) that $\mathrm{d}P/\mathrm{d}\lambda$ is positive for $\lambda = 1$ and also for sufficiently large values of λ. In order that P may rise to a maximum and then fall as λ increases, $\mathrm{d}P/\mathrm{d}\lambda$ must become negative for some value of $\lambda > 1$. This implies that for some value of $x > 1$ we must have

$$f(x) > \Gamma,$$

where

$$f(x) = \frac{x^3-7}{x^4+5x}. \tag{11·6}$$

This is possible only if Γ is less than the maximum value which $f(x)$ can attain. Solving $\mathrm{d}f/\mathrm{d}x = 0$, we see that $f(x)$ has a maximum value of 0·21 (approximately) when $x = 3·39$, i.e. when $\lambda = 1·84$. Hence, if the material of the spherical shell has a stored-energy function of the Mooney form with $\Gamma > 0·21$, the pressure will increase monotonically with inflation. If $\Gamma < 0·21$, an initial rise in pressure will be followed by a fall as the inflation proceeds, before the pressure commences to rise again at high degrees of inflation.

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